MAT 762, Algebraic Topology, Fall 2013

Homework Assignment 1

Problem 1.

(a) Show that $H_n(\mathbb{R}^m, \mathbb{R}^m - \{0\})$ is in general not isomorphic to $\widetilde{H}_n(\mathbb{R}^m/(\mathbb{R}^m - \{0\}))$.

(b) Compute the reduced singular homology of the Sierpiński space, i.e., of the two-point space in which exactly one of the two points forms an open set.

Hint. In (a), describe the space $\mathbb{R}^m/(\mathbb{R}^m - \{0\})$ (for $m > 0$) explicitly and observe that it is independent of m. In (b), show that the Sierpiński space is contractible.

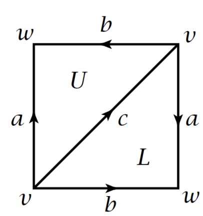
Problem 2. Compute the homology groups $H_n(\mathbb{RP}^m; \mathbb{Z}_2)$ by using the short exact sequence

$$0 \longrightarrow C_*(\mathbb{RP}^m; \mathbb{Z}_2) \xrightarrow{\tau} C_*(S^m; \mathbb{Z}_2) \xrightarrow{\pi} C_*(\mathbb{RP}^m; \mathbb{Z}_2) \longrightarrow 0\,,$$

where τ is the map which sends a singular simplex to the formal sum of its two lifts in S^m, and π is the map induced by the projection $S^m \to \mathbb{RP}^m$. You may assume as known that $H_n(S^m; \mathbb{Z}_2)$ (for $m > 0$) is equal to $\mathbb{Z}_2$ if $n = m$ or $n = 0$, and equal to zero otherwise.

Problem 3. A Δ-*complex* is a topological space X together with a collection $\mathcal{C}$ of singular simplices $\sigma_\alpha\colon \Delta^n \to X$ such that X is the disjoint union of the $\sigma_\alpha(\mathrm{int}(\Delta^n))$, and such that a set $U \subset X$ is open iff $\sigma_\alpha^{-1}(U)$ is open in Δ^n for each α. One further requires that each σ_α is injective on $\mathrm{int}(\Delta^n)$, and that every $(n-1)$-simplex which can be obtained from an n-simplex of $\mathcal{C}$ by composing with a face map ι_i^n is again an element of $\mathcal{C}$. For a Δ-complex X, one defines the *simplicial homology* as the homology of the subcomplex of $C_*(X)$ spanned by the simplices of $\mathcal{C}$. Compute the simplicial homology of the following Δ-complex:

$\mathbb{RP}^2$:



Problem 4. Show that for an R-module P, the following are equivalent:

(a) P is projective (i.e., has the lifting property).

(b) Every short exact sequence $0 \to A \to B \to P \to 0$ splits.

(c) There is an R-module K such that $P \oplus K$ is free.

(d) The functor $h^P := \mathrm{Hom}_R(P, -)$ is exact.

Hint. Prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

This homework is due on Tuesday, September 17, 2013.