MAT 762, Algebraic Topology, Fall 2013

Homework Assignment 3

Problem 1. Let $\Delta^n := [e_0, \dots, e_n]$ be the standard n-simplex and $v_i, w_i \subset \Delta^n \times [0,1]$ the points $v_i := (e_i, 0)$ and $w_i := (e_i, 1)$, and let

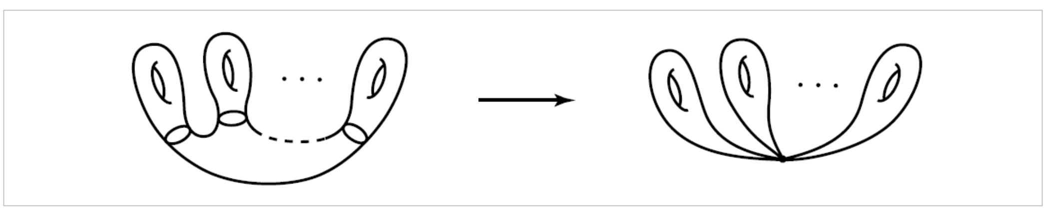
$$\iota_{v_0,\dots,v_i,w_n,\dots,w_i} \colon \Delta^{n+1} \longrightarrow \Delta^n \times [0,1]$$

be the affine map which sends each vertex of $\Delta^{n+1} = [e_0, \dots, e_{n+1}]$ to the corresponding vertex of the simplex $[v_0, \dots, v_i, w_n, \dots, w_i] \subset \Delta^n \times [0,1]$. For a space X, define $h_n \colon C_n(X) \to C_{n+1}(X)$ to be the linear map given by

$$h_n(\sigma) := \sum_{i=0}^{n} (-1)^i \epsilon_{n-i} (\sigma \pi \iota_{v_0,\dots,v_i,w_n,\dots,w_i})$$

where $\pi \colon \Delta^n \times [0,1] \to \Delta^n$ is the projection onto the first factor, and $\epsilon_k := (-1)^{k(k+1)/2}$. Show that $\partial h + h\partial = \rho - \mathrm{id}$ where $\rho(\sigma)(t_0, t_1, \dots, t_n) := \epsilon_n \sigma(t_n, \dots, t_1, t_0)$.

Problem 2. (Hatcher, Section 3.2, Exercise 1) Assuming as known the cup product structure on the torus $S^1 \times S^1$, compute the cup product structure in $H^*(M_g)$ for M_g the closed orientable surface of genus g by using the quotient map from M_g to a wedge sum of g tori, shown below.



Remark. Use that the homology of M_g is as described in Example 2.36 on page 141.

Problem 3. Let $n \geq 3$.

(a) Show that if $f \colon S^n \to S^{n-1}$ is a continuous map satisfying $f(-z) = -f(z)$ for all $z \in S^n$, then the induced map $g \colon \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^{n-1}$ sends the generator of $\pi_1(\mathbb{R}\mathrm{P}^n)$ to the generator of $\pi_1(\mathbb{R}\mathrm{P}^{n-1})$.

(b) Use the naturality of the isomorphisms

$$H^1(\mathbb{R}\mathrm{P}^k; \mathbb{Z}_2) \cong \mathrm{Hom}(H_1(\mathbb{R}\mathrm{P}^k), \mathbb{Z}_2) \cong \mathrm{Hom}(\pi_1(\mathbb{R}\mathrm{P}^k), \mathbb{Z}_2)$$

to show that the map g of part (a) sends the generator of $H^1(\mathbb{R}\mathrm{P}^{n-1}; \mathbb{Z}_2)$ to the generator of $H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$.

(c) Use the ring structure on $H^*(\mathbb{R}\mathrm{P}^k; \mathbb{Z}_2)$ to conclude that there cannot be a continuous map $f \colon S^n \to S^{n-1}$ satisfying $f(-z) = -f(z)$ for all $z \in S^n$.

Problem 4.

(a) Use acyclic models to prove that the following diagram commutes up to homotopy

$$\begin{tikzcd} C_*(X \times Y) \arrow[r, "t_*"] \arrow[d, "A"'] & C_*(Y \times X) \arrow[d, "A"'] \\ C_*(X) \otimes C_*(Y) \arrow[r, "\tau"] & C_*(Y) \otimes C_*(X) \end{tikzcd}$$

where A is the Alexander-Whitney map, $t\colon X \times Y \to Y \times X$ is the homeomorphism which sends a point (x, y) to the point (y, x), and τ is the isomorphism of chain complexes which sends $c \otimes c' \in C_i(X) \otimes C_j(Y)$ to $(-1)^{ij} c' \otimes c \in C_j(Y) \otimes C_i(X)$.

(b) Use part (a) to prove $t^*([\beta] \times [\alpha]) = (-1)^{ij}[\alpha] \times [\beta]$ for all $[\alpha] \in H^i(X)$ and $[\beta] \in H^j(Y)$.

(c) Conclude that $[\beta] \cup [\alpha] = (-1)^{ij}[\alpha] \cup [\beta]$ for all $[\alpha] \in H^i(X)$ and $[\beta] \in H^j(X)$.

This homework is due on Thursday, October 17, 2013.