MAT 762, Algebraic Topology, Fall 2013

Homework Assignment 4

Problem 1. Show that if $\{C_\alpha, f_{\alpha\beta}\}$ is a directed system of chain complexes of abelian groups, with the maps $f_{\alpha\beta}\colon C_\alpha \to C_\beta$ chain maps, then $H_n(\varinjlim C_\alpha) = \varinjlim H_n(C_\alpha)$.

Problem 2. A symplectic vector space is a vector space V equipped with a nondegenerate bilinear form ω satisfying $\omega(v, v) = 0$ for all $v \in V$.

(a) Show that every symplectic vector space $V \neq 0$ has a decomposition $V = U \oplus U^\perp$ where U is 2-dimensional and $U^\perp$ is orthogonal to U with respect to ω.

(b) Show that every finite-dimensional symplectic vector space is even-dimensional.

(c) Use Poincaré duality to show that if M is a closed, oriented, connected m-manifold, then the intersection pairing

$$\cdot\colon H_i(M;\mathbb{Q}) \otimes H_{m-i}(M;\mathbb{Q}) \longrightarrow H_0(M;\mathbb{Q}) \cong \mathbb{Q}$$

defined by $c \cdot c' := D_M(D_M^{-1}(c) \cup D_M^{-1}(c'))$ is nondegenerate.

(d) Show that the nth Betti number of a closed, oriented $2n$-manifold is even if n is odd.

Problem 3. Using cellular homology, one can see that

$$H_*(\mathbb{C}\mathrm{P}^n) \cong \mathbb{Z}e_0 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_4 \oplus \ldots \oplus \mathbb{Z}e_{2n}$$

where $e_i \in H_i(\mathbb{C}\mathrm{P}^n)$.

(a) Use the nondegeneracy of the cup product (cf. Problem 2) to show that $e_i^* \cup e_{2n-i}^* = \pm e_{2n}^*$, where $e_i^* \in H^i(\mathbb{C}\mathrm{P}^n) \cong H_i(\mathbb{C}\mathrm{P}^n)^*$ denotes the generator dual to e_i, for i even and $\leq 2n$.

(b) Use the map $\mathbb{C}\mathrm{P}^{i+j} \to \mathbb{C}\mathrm{P}^n$ and part (a) to show that $e_i^* \cup e_j^* = \pm e_{i+j}^*$, for i, j even and $i + j \leq 2n$.

(c) Show that the integer cohomology ring of $\mathbb{C}\mathrm{P}^n$ is isomorphic to $\mathbb{Z}[\alpha]/(\alpha^{n+1})$ where α is the generator e_2^* of $H^2(\mathbb{C}\mathrm{P}^n)$.

Problem 4. (Hatcher, Section 3.3, Exercise 10) Show that if $f\colon M \to N$ is a degree 1 map of connected closed oriented manifolds, then the induced map $f_*\colon \pi_1(M) \to \pi_1(N)$ is surjective.

This homework is due on Tuesday, November 12, 2013.