MAT 762, Algebraic Topology, Fall 2013

Homework Assignment 5

Problem 1. (Hatcher, Section 3.3, Exercise 11) If M_g denotes the closed orientable surface of genus g, show that continuous degree 1 maps $M_g \to M_h$ exist iff $g \geq h$.
Hint. Use Problem 4 of the previous homework assignment together with the fact that $\mathrm{Ab}(\pi_1(M_g)) = H_1(M_g) \cong \mathbb{Z}^{2g}$.

Problem 2. (Hatcher, Section 3.3, Exercise 12) As an algebraic application of the preceding problem, show that in a free group F with basis $x_1, \ldots, x_{2k}$, the product of commutators $[x_1, x_2] \cdots [x_{2k-1}, x_{2k}]$ is not equal to a product of fewer than g commutators $[v_i, w_i]$ of elements $v_i, w_i \in F$.
Hint. Recall that the 2-cell of M_k is attached by the product $[x_1, x_2] \cdots [x_{2k-1}, x_{2k}]$. From a relation $[x_1, x_2] \cdots [x_{2k-1}, x_{2k}] = [v_1, w_1] \cdots [v_j, w_j]$ in F, construct a degree 1 map $M_j \to M_k$.

Problem 3.

(a) Let K be a 2-knot (i.e., an embedded 2-sphere in S^4). Use Alexander duality to compute $H_i(S^4 \setminus K)$ for all i.

(b) For $n > 1$, let $M \subset S^n$ be a connected embedded nonempty closed orientable $(n-1)$-manifold. Use Alexander duality to show that $S^n \setminus M$ has two path components.

Problem 4.

(a) Show that if $f \in C^\infty(\mathbb{R}^n)$ is a smooth $\mathbb{R}$-valued function on $\mathbb{R}^n$, then there are smooth $\mathbb{R}$-valued functions $g_1, \ldots, g_n \in C^\infty(\mathbb{R}^n)$ such that $f(x) = f(0) + \sum_{i=1}^n x_i g_i(x)$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

(b) Let $(\partial/\partial x_i)_0 \colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$ denote the map defined by $(\partial/\partial x_i)_0(f) := (\partial f/\partial x_i)(0)$. Show that the maps $(\partial/\partial x_1)_0, \ldots, (\partial/\partial x_n)_0$ form a basis for the $\mathbb{R}$-vector space

$$T_0\mathbb{R}^n := \{\mathbb{R}\text{-linear maps } v \colon C^\infty(\mathbb{R}^n) \to \mathbb{R} \text{ satisfying } v(fg) = v(f)g(0) + f(0)v(g)\}.$$

Hint. In (a), use that $f(x) - f(0) = \int_0^1 (\frac{df}{dt})(tx)dt$ and that $(\frac{df}{dt})(tx) = \sum_{i=1}^n x_i (\frac{\partial f}{\partial x_i})(tx)$.

This homework is due on Tuesday, December 3, 2013.