CATEGORIFICATION OF THE COLORED JONES POLYNOMIAL AND RASMUSSEN INVARIANT OF LINKS

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Abstract. We define a family of formal Khovanov brackets of a colored link depending on two parameters. The isomorphism classes of these brackets are invariants of framed colored links. The Bar–Natan functors applied to these brackets produce Khovanov and Lee homology theories categorifying the colored Jones polynomial. Further, we study conditions under which framed colored link cobordisms induce chain transformations between our formal brackets. We conjecture that, for special choice of parameters, Khovanov and Lee homology theories of colored links are functorial (up to sign). Finally, we extend the Rasmussen invariant to links and give examples, where this invariant is a stronger obstruction to sliceness than the multivariable Levine–Tristram signature.

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Introduction

In [6], Khovanov constructed a bigraded chain complex, whose Euler characteristic is the Jones polynomial and whose chain equivalence class is a link invariant. In particular, the bigraded homology group, known as Khovanov homology, is a link invariant. Bar–Natan [2] and the second author [12] showed that Khovanov homology is strictly stronger than the Jones polynomial. Furthermore, Khovanov homology is functorial with respect to link cobordisms smoothly embedded in $\mathbb{R}^4$.

In [8], Lee modified Khovanov construction and made it more accessible for calculations. The generators of Lee homology are known explicitly. The middle topological degree of the two generators of Lee homology is a new knot invariant introduced by Rasmussen [10]. Rasmussen used it to give a combinatorial proof of the Milnor conjecture. Note that this conjecture was previously accessible only via gauge theory – instanton Donaldson invariants, Seiberg–Witten theory or Ozsváth–Szabó knot Floer homology. Viewing Khovanov theory as a combinatorial counterpart of

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the knot Floer homology of Ozsváth and Szabó, one can expect that the categorification of quantum 3–manifold invariants will provide a combinatorial approach to Heegaard Floer homology.

The first step in this direction is a categorification of the colored Jones polynomial. In [7], Khovanov made two proposals for such a homology theory, based on two natural normalizations of the colored Jones polynomial. Unfortunately, the first homology theory categorifying the colored Jones polynomial is defined over $\mathbb{Z}/2\mathbb{Z}$ and the second one, for the reduced Jones polynomial, works for knots only. In this paper, we develop both Lee and Khovanov homology theories of colored links over $\mathbb{Z}[1/2]$. To do this, we explore the ideas of Bar–Natan [3], who regard these theories as just different functors applied to the *formal Khovanov bracket.* A similar approach to constructing new homology theories over $\mathbb{Z}/2\mathbb{Z}$ for colored links was independently proposed by Mackaay and Turner [9].

Our main results are summarized in the next subsection.

0.1. **Main results.** Let $\mathbf{n} = \{n_1, n_2, ..., n_l\}$ be a finite sequence of natural numbers. Let $L_\mathbf{n}$ be an oriented framed colored link of l components, where n_i is the color of the i–th component, and $D_\mathbf{n}$ be its diagram in blackboard framing.

In Section 2 we define the formal Khovanov bracket $[\![D_\mathbf{n}]\!]_{\alpha,\beta}$ of the colored link $L_\mathbf{n}$ as an object of $\mathrm{Kom}(\mathrm{Mat}(\mathrm{Kob}_{/h}))$. Here $\mathrm{Kom}(\mathrm{Mat}(\mathrm{Kob}_{/h}))$ is the category of formal complexes over a 'matrix extension' of the category $\mathrm{Kob}_{/h}$, where Bar–Natan's formal brackets of links belong to (see Section 1).

$\mathrm{Kob}_{/h}$ is itself a homotopy category of complexes, so we may think of $[\![D_\mathbf{n}]\!]_{\alpha,\beta}$ as a 'complex of complexes'. The subscripts α and β are two integer parameters which enter in the definition of the differential of $[\![D_\mathbf{n}]\!]_{\alpha,\beta}$.

We show that

Theorem 1. *For any α and β, the isomorphism class of the complex $[\![D_\mathbf{n}]\!]_{\alpha,\beta}$ is an invariant of the colored framed oriented link $L_\mathbf{n}$.*

Let $\mathcal{A}$ be the category of $\mathbb{Z}[1/2]$–modules. By applying the Khovanov functor $\mathcal{F}_{\mathrm{Kh}}$ and the Lee functor $\mathcal{F}_{\mathrm{Lee}}$ to the formal bracket, we get homology theories over $\mathrm{Kom}_{/h}(\mathcal{A})$.

Corollary 2. *The total graded Euler characteristic of $\mathcal{F}_{\mathrm{Kh}}([\![D_\mathbf{n}]\!]_{1,0})$ is equal to the colored Jones polynomial of $L_\mathbf{n}$.*

The precise definition of the total graded Euler characteristic will be given in Subsection 2.3. Note that Khovanov's [7] categorification of the colored Jones polynomial is a variant of $\mathcal{F}_{\mathrm{Kh}}([\![D_\mathbf{n}]\!]_{1,0})$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

In Section 3 we study movie presentations of framed cobordisms, where by a framed cobordism we mean a compact smooth oriented surface which is properly embedded in $\mathbb{R}^3 \times I$ and equipped with a trivialization of its normal bundle in $\mathbb{R}^3 \times I$, and which connects a framed link in $\mathbb{R}^3 \times \{0\}$ to a framed link in $\mathbb{R}^3 \times \{1\}$. We extend the Carter–Saito movie moves [4] to the setting of framed cobordisms.

Theorem 3. *Two movies present isotopic framed cobordisms if and only if there is a sequence of modified Carter–Saito moves and additional moves depicted in Figure 4 that takes one movie to the other.*

A colored framed cobordism is a framed cobordism together with a coloring of its connectivity components by natural numbers. Colored framed cobordisms have movie presentations whose stills are colored framed link diagrams. Let $\mathcal{C}\mathbf{ob}^4_f$ be the category whose objects are colored framed link diagrams and whose morphisms are movie presentations of colored framed cobordisms. In Sections 4 and 5 we show that $\mathcal{F}_{\rm Kh}([\![D_n]\!]_{0,1})$ and $\mathcal{F}_{\rm Lee}([\![D_n]\!]_{1,1})$ extend to functors $\mathcal{F}_{\rm Kh} \circ \mathcal{K}\mathbf{h}_{0,1}$ and $\mathcal{F}_{\rm Lee} \circ \mathcal{K}\mathbf{h}_{1,1}$, respectively, from $\mathcal{C}\mathbf{ob}^4_f$ to the category of complexes over $\mathcal{A}$. More precisely,

Theorem 4. *The functors $\mathcal{F}_{\rm Kh}\circ\mathcal{K}\mathbf{h}_{0,1}$ and $\mathcal{F}_{\rm Lee}\circ\mathcal{K}\mathbf{h}_{1,1}$ from $\mathcal{C}\mathbf{ob}^4_f$ to ${\rm Kom}({\rm Kom}_{/h}(\mathcal{A}))$ are well–defined.*

Let $\mathcal{C}\mathbf{ob}^4_{f/i}$ be the quotient of $\mathcal{C}\mathbf{ob}^4_f$ by framed Carter–Saito movie movies, and ${\rm Kom}_{/h}({\rm Kom}_{/h}(\mathcal{A}))_{/\pm}$ be the projectivization of ${\rm Kom}_{/h}({\rm Kom}_{/h}(\mathcal{A}))$, where each morphism is identified with its negative. We expect

Conjecture 5. *The functors $\mathcal{F}_{\rm Kh} \circ \mathcal{K}\mathbf{h}_{0,1}$ and $\mathcal{F}_{\rm Lee} \circ \mathcal{K}\mathbf{h}_{1,1}$ descend to functors $\mathcal{C}\mathbf{ob}^4_{f/i} \to {\rm Kom}_{/h}({\rm Kom}_{/h}(\mathcal{A}))_{/\pm}$.*

Finally, we extend the definition of the Rasmussen invariant to links and study its properties. We show that in some cases the Rasmussen invariant of links is a stronger obstruction to sliceness than the multivariable Levine–Tristram signature defined by Cimasoni and Florens [5].

Another interesting application of the Rasmussen invariant of links was found by Baader [1]. He used the Rasmussen invariant to define a quasimorphism on the braid group and to estimate the torsion length for alternating braids.

0.2. **Plan of the paper.** In Section 1 we recall the Bar–Natan construction. Then we define the formal Khovanov bracket of colored links. In Section 3 we study framed cobordisms and their movie presentations. Further, we construct maps between our formal brackets induced by colored framed link cobordisms. The last section is devoted to the Rasmussen invariant of links.

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1. Bar–Natan's construction

In [3], Bar–Natan defined the formal Khovanov bracket $[\![\cdot]\!]$ for any link (or tangle) in such a way that the Khovanov and Lee's homology theories can be reconstructed from $[\![\cdot]\!]$. In this section we briefly recall the Bar–Natan's construction.

1.1. **Formal Khovanov bracket.** Suppose we have a generic diagram D of an oriented link L in S^3 with c crossings. There is a cube of resolutions associated to D (compare [2]). The vertices of the cube correspond to the configurations of circles obtained after smoothing of all crossings in D. For any crossing, two different smoothings are allowed: the 0– and the 1–smoothing. Therefore, we have 2^c vertices. After numbering the crossings of D, we can label the vertices of the cube by c–letter strings of 0's and 1's, specifying the smoothing chosen at each crossing. The cube is skewered along its main diagonal, from 00...0 to 11...1. The number of 1 in the labeling of a vertex is equal to its 'height' k. The cube is displayed in such a way that the vertices of height k project down to the point $r := k - c_-$ (see Figure 1).

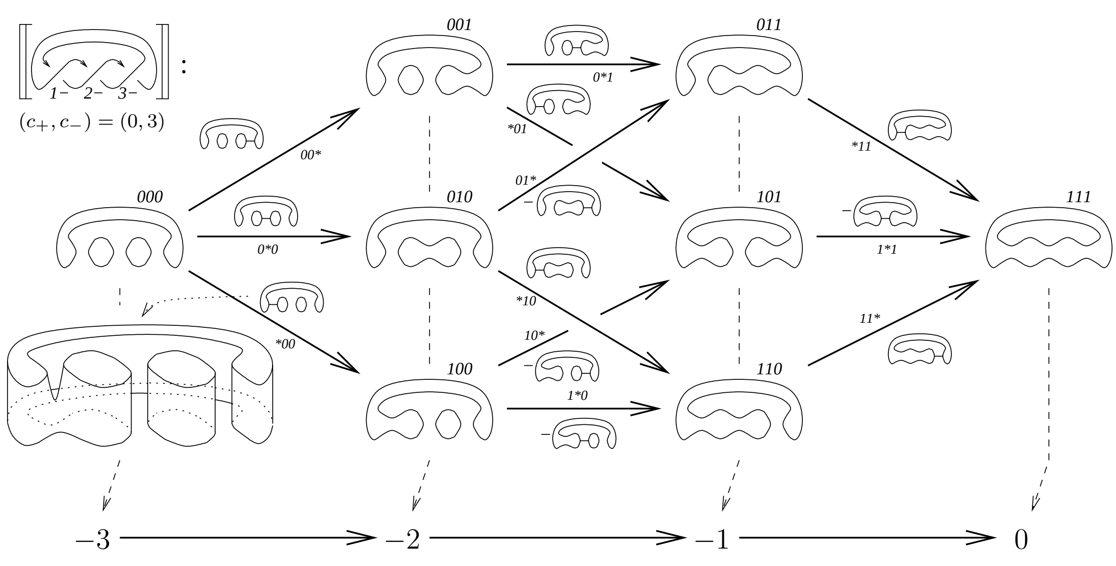


Figure 1 *The cube of resolutions for the trefoil*

Two vertices of the cube are connected by an edge if their labelings differ by one letter. The edges are directed (from the vertex where this letter is 0 to the vertex where it is 1). The edges correspond to cobordisms from the tail configuration of circles to the head configuration (compare Figure 1).

Bar–Natan proposed to interpret the cube of resolutions denoted $[\![D]\!]$ as a complex, where all smoothings are considered as spaces and all cobordisms as maps. The rth chain space $[\![D]\!]^r$ of the complex $[\![D]\!]$ is a formal direct sum of the $\frac{c!}{k!(c-k)!}$ "spaces" at height k in the cube and the sum of "maps" with tails at height k defines the rth differential.

More precisely, $[\![D]\!]$ is considered as an object of $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3))$. Here $\mathcal{C}ob^3$ is the additive category whose objects are circle configurations (smoothings) and morphisms are 2–cobordisms between such smoothings. For any additive category $\mathcal{C}$, $\mathrm{Mat}(\mathcal{C})$ is the category whose objects are formal direct sums of objects of $\mathcal{C}$ and whose composition law is modeled on the matrix multiplication. $\mathrm{Kom}(\mathcal{C})$ is the category of complexes over $\mathcal{C}$, where objects are chains of finite length and morphisms are chain transformations.

Let us impose some local relations in $\mathcal{C}ob^3$: (S) any cobordism containing a closed sphere as a connected component is set to be zero; (T) any closed torus can be removed from a cobordisms at cost of the factor 2; (4Tu) the four tube relation defined in [3]. The neck cutting relation drawn below is a special form of the 4Tu. If 2 is invertible, we can use this relation to cut any tube inside a cobordism.

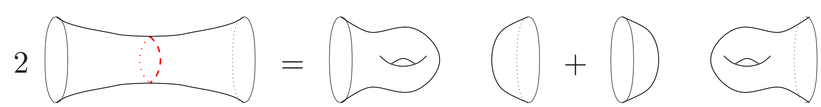


Figure 2 *The neck cutting relation*

We denote the quotient of $\mathcal{C}ob^3$ by these relations $\mathcal{C}ob^3_{/l}$ and consider $[\![D]\!]$ as an object of $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_{/l})$. We set $\mathrm{Kob} := \mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_{/l})$

Theorem 1.1 (Bar–Natan)**.** *The homotopy type of* $[\![D]\!]$ *is an invariant of* L*.*

In [3], Bar–Natan constructed explicit homotopies between complexes related by the three Reidemeister moves.

1.2. **Topological grading.** A pre–additive category $\mathcal{C}$ is called graded if it has the following additional properties. Its morphism sets are graded Abelian groups, and the degree is additive under composition of morphisms. Moreover, there is a $\mathbb{Z}$–action $(m, \mathcal{O}) \mapsto \mathcal{O}\{m\}$ on the objects $\mathcal{O}$ of $\mathcal{C}$, which shifts the gradings of the morphisms, but such that $\mathrm{Mor}(\mathcal{O}_1\{m_1\}, \mathcal{O}_2\{m_2\}) = \mathrm{Mor}(\mathcal{O}_1, \mathcal{O}_2)$ as plain Abelian groups.

Bar–Natan [3] observed that $\mathcal{C}ob^3_{/l}$ can be transformed into a graded category by introducing artificial objects $\mathcal{O}\{m\}$ for every $m \in \mathbb{Z}$ and every object $\mathcal{O} \in$

$\mathrm{Obj}(\mathcal{C}ob^3_{/l})$, and by defining the degree of a cobordism $S \in \mathrm{Mor}(\mathcal{C}ob^3_{/l})$ to be its Euler characteristic. In what follows, we will denote by $\mathcal{C}ob^3_{/l}$ this graded category, and we will refer to its grading as the topological grading. Note that the topological grading of $\mathcal{C}ob^3_{/l}$ induces topological gradings on $\mathrm{Mat}(\mathcal{C}ob^3_{/l})$ and $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_{/l}))$.

1.3. **Functoriality.** A link cobordism is is a compact oriented surface which is smoothly and properly embedded in $\mathbb{R}^3 \times I$ and connects a link in $\mathbb{R}^3 \times \{0\}$ to a link in $\mathbb{R}^3 \times \{1\}$. Splitting cobordisms into pieces by planes $\mathbb{R}^3 \times \{t\}$, $0 \leq t \leq 1$, and projecting down to the plane, we can view them as a sequence of link diagrams or a movie of diagrams. Altering t, we can assume that any two consecutive diagrams in the movie differ by one of the following transformations — a Reidemeister move, a cap or a cup, or a saddle. It was shown in [4] that two such movies present isotopic cobordisms if and only if they can be related by a finite sequence of Carter–Saito movie moves.

Let $\mathcal{C}ob^4$ be the category whose objects are oriented link diagrams, and whose morphisms are movie presentations of cobordisms between links described by such diagrams. Let $\mathcal{C}ob^4_{/i}$ be the quotient of $\mathcal{C}ob^4$ by Carter–Saito movie moves.

The formal Khovanov bracket descends to a functor from $\mathrm{Kh} : \mathcal{C}ob^4 \to \mathrm{Kob}$. On the objects, $\mathrm{Kh}(D)$ is defined as the complex $\mathrm{Kh}^r(D) := [\![D]\!]\{r + c_+ - c_-\}$, whose differentials are the same as those of $[\![D]\!]$. Note that all differentials in $\mathrm{Kh}(D)$ are of topological degree zero. Moreover, it follows from the proof of Theorem 1.1 that the graded homotopy type of $\mathrm{Kh}(D)$ is a link invariant (cf. [3, Theorem 3]).

On the generating morphisms Kh is defined as follows: For the Reidemeister moves we take the chain homotopies constructed in [3] for the proof of Theorem 1.1. For the cup, cap or the saddle, we take the natural chain transformations given by the corresponding cobordisms.

Let $\mathrm{Kob}_{/h}$ be the category Kob modulo homotopies, i.e. it has the same objects as Kob, but homotopic morphisms in Kob are identified. Let $\mathrm{Kob}_{/\pm h}$ be the projectivization of $\mathrm{Kob}_{/h}$.

Theorem 1.2 (Bar–Natan)**.** Kh *descends to a functor* $\mathrm{Kh} : \mathcal{C}ob^4_{/i} \to \mathrm{Kob}_{/\pm h}$.

By the Carter–Saito theorem [4], movie presentations of isotopic cobordisms are related by 15 movie moves. Bar–Natan proved that the morphisms in Kob induced by these movies moves are homotopic up to signs.

1.4. **Khovanov and Lee's theories.** Any functor from $\mathcal{C}ob^3_{/l}$ to an Abelian category $\mathcal{A}$ extends to a functor $\mathcal{F} : \mathrm{Kob} \to \mathrm{Kom}(\mathcal{A})$ providing a homology theory. If in addition $\mathcal{A}$ is graded, and $\mathcal{F}$ is degree–respecting, then the homology is a graded invariant of a link.

1.4.1. *Khovanov functor.* Let $\mathcal{O} \in \mathrm{Obj}(\mathcal{C}ob^3_{/l})$ and $\mathbb{Z}_{(2)} = \mathbb{Z}[1/2]$. We put

$$\mathcal{F}_{\mathrm{Kh}}(\mathcal{O}) := \mathbb{Z}_{(2)} \otimes_{\mathbb{Z}} \mathrm{Mor}(\emptyset, \mathcal{O})/\mathrm{Rel}_{g>1}$$

where by $\mathrm{Rel}_{g>1}$ all cobordisms of genus greater than 1 are set to be zero. With a circle, $\mathcal{F}_{\mathrm{Kh}}$ associates the $\mathbb{Z}_{(2)}$–module of rank 2 generated by $v_+ :=$  and by $v_- := \frac{1}{2}$ . The neck cutting relation allows to identify the differentials in this theory with the ones given by Khovanov [6] (compare Exercise 9.3 in [3]). With the natural choice of grading on $\mathbb{Z}_{(2)}$–modules ($\deg(v_+) = 1$, $\deg(v_-) = -1$), the functor $\mathcal{F}_{\mathrm{Kh}}$ is degree–respecting.

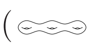
Hence $\mathcal{F}_{\mathrm{Kh}}(\mathrm{Kh}(D))$ is a complex in the category of graded $\mathbb{Z}_{(2)}$–modules. We define its graded Euler characteristic $\chi(\mathcal{F}_{\mathrm{Kh}}(\mathrm{Kh}(D))) \in \mathbb{Z}[q, q^{-1}]$ by

$$\chi(\mathcal{F}_{\mathrm{Kh}}(\mathrm{Kh}(D))) := \sum_{r,j} (-1)^r q^j \dim_{\mathbb{Q}}(M^{r,j}(D) \otimes_{\mathbb{Z}_{(2)}} \mathbb{Q}),$$

where $M^{r,j}(D)$ denotes the homogeneous component of degree j of the graded $\mathbb{Z}_{(2)}$–module $\mathcal{F}_{\mathrm{Kh}}(\mathrm{Kh}^r(D))$. It was shown in [6] that $\chi(\mathcal{F}_{\mathrm{Kh}}(\mathrm{Kh}(D)))$ is equal to the Jones polynomial of the link represented by the diagram D.

1.4.2. *Lee's functor.* Let us put

$$\mathcal{F}_{\mathrm{Lee}}(\mathcal{O}) := \mathbb{Z}_{(2)} \otimes_{\mathbb{Z}} \mathrm{Mor}(\emptyset, \mathcal{O})/(\ \text{[genus 3 surface]} = 8)$$



where the relation set the morphism given by the genus 3 surface without boundary to be 8. Here the same rank 2 module is associated to the circle. But the differentials Δ and m are given by the Lee's formulas [8]:

$$\Delta : \begin{cases} a \mapsto a \otimes a \\ b \mapsto b \otimes b \end{cases} \qquad m_2 : \begin{cases} a \otimes a \mapsto 2a & b \otimes b \mapsto -2b \\ a \otimes b \mapsto 0 & b \otimes a \mapsto 0, \end{cases} \tag{1}$$

where $a := v_+ + v_-$ and $b := v_+ - v_-$. The Lee functor is not degree–respecting.

2. Formal Khovanov bracket of a colored link

The aim of this section is to define the formal Khovanov bracket of a colored link. Our first approach is inspired by Khovanov [7]. Its modifications are necessary in order to get functoriality with respect to colored framed link cobordisms.

2.1. **Colored Jones polynomial.** Let $\mathbf{n} = \{n_1, n_2, ..., n_l\}$ be a finite sequence of natural numbers. Let $L_{\mathbf{n}}$ be an oriented framed l component link, whose i–th component is colored by the $(n_i + 1)$–dimensional irreducible representation of $\mathfrak{sl}_2$. Let $J(L^{\mathbf{n}})$ be the Jones polynomial of $\mathbf{n}$–cable of L. When forming the m–cable of a component K, we orient the stands by alternating the original and the opposite directions. More precisely, let us enumerate the strands from left to right from 1 to m. Then strand 1 is oriented in the same way as K, the strand 2 is oppositely oriented, etc.

The colored Jones polynomial is given by the following formula.

$$J_{\mathbf{n}}(L) = \sum_{\mathbf{k}=\mathbf{0}}^{\lfloor \mathbf{n}/2 \rfloor} (-1)^{|\mathbf{k}|} \begin{pmatrix} \mathbf{n}-\mathbf{k} \\ \mathbf{k} \end{pmatrix} J(L^{\mathbf{n}-2\mathbf{k}}) \tag{2}$$

where $|\mathbf{k}| = \sum_i k_i$, and

$$\begin{pmatrix} \mathbf{n}-\mathbf{k} \\ \mathbf{k} \end{pmatrix} = \prod_{i=1}^{l} \begin{pmatrix} n_i - k_i \\ k_i \end{pmatrix} .$$

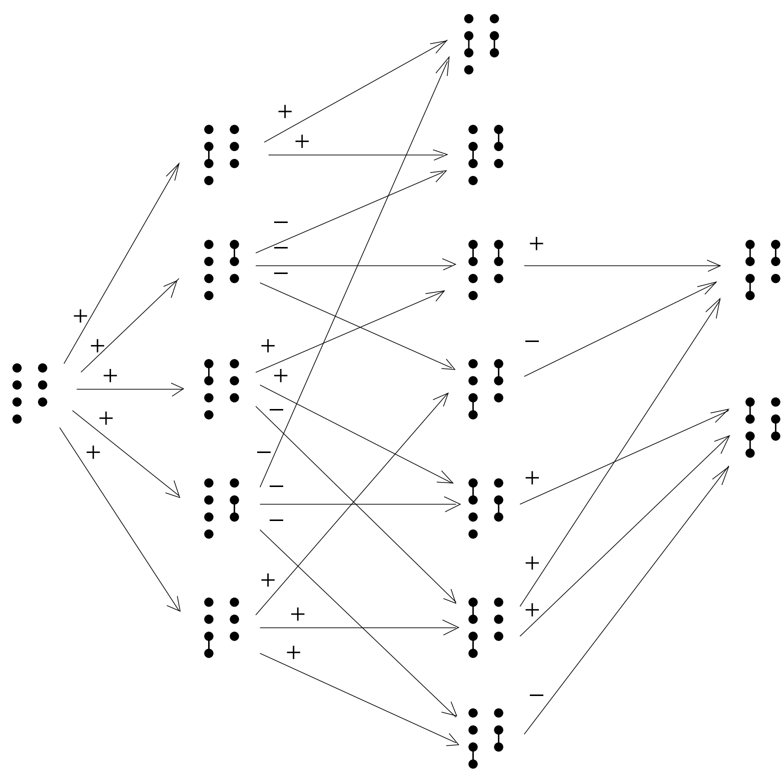


Figure 3 *The graph* $\Gamma_{4,3}$.

2.2. **Graph $\Gamma_{\mathbf{n}}$.** The binomial coefficient $\begin{pmatrix} n-k \\ k \end{pmatrix}$ equals the number of ways to select k pairs of neighbors from n dots placed on a line, such that each dot appears in at most one pair. Analogously, $\begin{pmatrix} \mathbf{n}-\mathbf{k} \\ \mathbf{k} \end{pmatrix}$ is the number of ways to select $\mathbf{k}$ pairs of neighbors on l lines. We will call these choices $\mathbf{k}$–pairings.

Let $\Gamma_{\mathbf{n}}$ be the graph, whose vertices correspond to $\mathbf{k}$–pairings. Two vertices of $\Gamma_{\mathbf{n}}$ are connected by an edge if the corresponding pairings can be related to each other by adding/removing one pair of neighboring points. The height of a vertex labeled by a $\mathbf{k}$–pairing is equal $|\mathbf{k}|$. The edges are directed towards increasing of heights (see Figure 3).

2.3. **Colored Khovanov bracket. First approach.** Let $L_{\mathbf{n}}$ be an oriented framed colored link as above and let $D_{\mathbf{n}}$ be its generic diagram in blackboard framing. Given $\Gamma_{\mathbf{n}}$ as above, we associate to it the formal Khovanov bracket $[\![D_{\mathbf{n}}]\!]$ of $L_{\mathbf{n}}$ regarded as an element of $\mathrm{Kom}(\mathrm{Mat}(\mathrm{Kob}_{/h}))$. The construction goes as follows.

At each vertex of $\Gamma_{\mathbf{n}}$ labeled by a $\mathbf{k}$–pairing we put the complex $\mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}}) \in \mathrm{Obj}(\mathrm{Kob}_{/h})$ defined in Subsection 1.3.

With an edge e of $\Gamma_{\mathbf{n}}$ connecting $\mathbf{k}$– and $\mathbf{k}'$–pairings, we associate a morphism $\mathrm{Kh}(e) : \mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}}) \to \mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}'})$ given by gluing of an annulus between the strands of the cable which form a pair in $\mathbf{k}'$, but not in $\mathbf{k}$. According to the definition of $\Gamma_{\mathbf{n}}$ there is only one such pair. Note that we view the complexes $\mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}})$ and $\mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}'})$ as objects of the homotopy category $\mathrm{Kob}_{/h}$, so that $\mathrm{Kh}(e)$ is a homotopy class of chain transformations. By Theorem 1.2 this homotopy class is well–defined up to sign. The sign of $\mathrm{Kh}(e)$ depends on the choice of the movie presentation for the annulus. We call this choice satisfactory if all squares of $\Gamma_{\mathbf{n}}$ anticommute. Note that by Theorem 1.2 the squares of $\Gamma_{\mathbf{n}}$ commute up to sign, because the cobordisms given by gluing of annuli in a different order are isotopic.

Given a satisfactory choice of signs, the result is a complex in $\mathrm{Kom}(\mathrm{Mat}(\mathrm{Kob}_{/h}))$, which we denote $[\![D_{\mathbf{n}}]\!]$. The i–th chain of $[\![D_{\mathbf{n}}]\!]$ is a formal direct sum of complexes at height i, i.e. $[\![D_{\mathbf{n}}]\!]^i := \oplus_{|\mathbf{k}|=i} \oplus_{\mathbf{s}\in\mathbf{k}} \mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}})$, where the notation $\mathbf{s} \in \mathbf{k}$ means that $\mathbf{s}$ is a $\mathbf{k}$–pairing. The i–th differential $d_i : [\![D_{\mathbf{n}}]\!]^i \to [\![D_{\mathbf{n}}]\!]^{i+1}$ is the formal sum of all morphisms $\mathrm{Kh}(e)$ corresponding to edges with tails at height i. Because the Euler characteristic of an annulus is zero, all $\mathrm{Kh}(e)$ have topological degree zero, and therefore $[\![D_{\mathbf{n}}]\!]$ inherits a topological grading from the topological gradings of the complexes $\mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}})$. Besides the topological grading, $[\![D_{\mathbf{n}}]\!]$ has two homological gradings, one corresponding to the differential d_i and one to the differentials of the complexes $\mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}})$. Note however that $[\![D_{\mathbf{n}}]\!]$ is not a bicomplex because the chain transformations $\mathrm{Kh}(e)$ are considered up to homotopy. It is an interesting problem whether one can construct a bicomplex, possibly by choosing suitable representatives for the homotopy classes $\mathrm{Kh}(e)$. If such a bicomplex exists, there should be a spectral sequence whose E_2 term is determined by $[\![D_{\mathbf{n}}]\!]$ and which converges to the homology of the total complex of that bicomplex.

We do not know how to form a bicomplex, but we can define a total graded Euler characteristic as follows. Let $[\![D_{\mathbf{n}}]\!]^{i,r} \in \mathrm{Obj}(\mathrm{Mat}(\mathcal{C}ob^3_{/l}))$ be the formal direct sum $[\![D_{\mathbf{n}}]\!]^{i,r} := \oplus_{|\mathbf{k}|=i} \oplus_{\mathbf{s}\in\mathbf{k}} \mathrm{Kh}^r(D^{\mathbf{n}-2\mathbf{k}})$, where $\mathrm{Kh}^r(D^{\mathbf{n}-2\mathbf{k}})$ denotes the r–th chain of the complex $\mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}})$. The functor $\mathcal{F}_{\mathrm{Kh}}$ maps $[\![D_{\mathbf{n}}]\!]^{i,r}$ to a graded $\mathbb{Z}_{(2)}$–module whose j–th homogeneous component we denote $M^{i,r,j}(D_{\mathbf{n}})$. The total graded Euler characteristic of $\mathcal{F}_{\mathrm{Kh}}([\![D_{\mathbf{n}}]\!])$ is defined by

$$\chi(\mathcal{F}_{\mathrm{Kh}}([\![D_{\mathbf{n}}]\!])) := \sum_{i,r,j} (-1)^{i+r} q^j \dim_{\mathbb{Q}}(M^{i,r,j}(D_{\mathbf{n}}) \otimes_{Z_{(2)}} \mathbb{Q}) \, .$$

To complete this subsection, we prove the following lemma, which shows that our construction of the colored Khovanov bracket is well–defined.

Lemma 2.1. *For any graph $\Gamma_{\mathbf{n}}$ there exists a satisfactory choice of signs making all squares anticommutative. Complexes defined with different satisfactory sign choices are isomorphic.*

Proof. Let us first show that we can make all squares commutative. We define a 1–cochain $\zeta \in C^1(\Gamma_{\mathbf{n}}, \mathbb{Z}/2\mathbb{Z})$ as follows. For any square $s \subset \Gamma_{\mathbf{n}}$, we put $\zeta(s) = 1$ if s is anticommutative and zero otherwise. We extend ζ by linearity to $\Gamma_{\mathbf{n}}$. Now we multiply any map $\mathrm{Kh}(e)$ by $(-1)^{\zeta(e)}$.

Note that ζ is well–defined, because there are no squares which are commutative and anticommutative simultaneously. In other words, the composition of maps induced by gluing of annuli is never zero. Indeed, let $\kappa : \mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}}) \to \mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}''})$ be a map induced by gluing $|\mathbf{k}| - |\mathbf{k}''|$ annuli. Let $\bar{\kappa} : \mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}''}) \to \mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}})$ denote the map induced by the same annuli "turned upside down". In the composition $\kappa\bar{\kappa}$, every annulus of κ is glued with the corresponding annulus of $\bar{\kappa}$, such that the result is a torus. Hence $\kappa\bar{\kappa}$ is induced by the union of $D^{\mathbf{n}-2\mathbf{k}''} \times [0,1]$ with a collection of $|k| - |k''|$ tori. After isotopy, we can assume that these tori lie in $\mathbb{R}^3 \times \{1/2\}$. In $\mathbb{R}^3 \times \{1/2\}$, the tori may be linked with $D^{\mathbf{n}-2\mathbf{k}''} \times \{1/2\}$, but if we consider $2^{|\mathbf{k}|-|\mathbf{k}''|}\kappa\bar{\kappa}$ instead of $\kappa\bar{\kappa}$, we can apply the neck cutting relation to obtain unlinked tori. It follows from the (T) relation that $2^{|\mathbf{k}|-|\mathbf{k}''|}\kappa\bar{\kappa}$ is equal to $4^{|\mathbf{k}|-|\mathbf{k}''|}$ times the identity morphism of $\mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}''})$, and hence κ is nonzero.

Given a complex with all squares commutative, we can make them anticommutative as follows. We multiply $\mathrm{Kh}(e)$ with (-1) to the power number of pairings to the right and above of the unique pairing in $\mathbf{k}' \setminus \mathbf{k}$. These signs are shown in Figure 3.

Given two satisfactory sign choices, the corresponding 1–cochains ζ and ζ' coincide on all squares, i.e. $\zeta - \zeta' = \delta\gamma$ with $\gamma \in C^0(\Gamma_{\mathbf{n}}, \mathbb{Z}/2\mathbb{Z})$. For any edge e with boundary $s - s'$, we have $\zeta(e) - \zeta'(e) = \gamma(s) - \gamma(s')$. Therefore, $(-1)^\gamma$ times the identity map defines an isomorphism between the corresponding complexes. $\square$

Remark 1. Lemma 2.1 shows that the categorification of the colored Jones polynomial in [7] can be defined over integers.

2.4. **Colored Khovanov bracket.** In the following, we work with coefficients in $\mathbb{Z}_{(2)} = \mathbb{Z}[1/2]$. That is, we replace the category $\mathcal{C}ob^3$ of Section 1.1 by the category which has the same objects as $\mathcal{C}ob^3$ but whose morphisms are formal $\mathbb{Z}_{(2)}$ linear combinations of cobordisms.

Let us generalize the definition of $[\![D_\mathbf{n}]\!]$ as follows. As before, we put $\mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}})$ at vertices of $\Gamma_\mathbf{n}$ labeled by $\mathbf{k}$–pairings. But we modify the maps associated to edges of $\Gamma_\mathbf{n}$. With an edge e connecting $\mathbf{k}$– and $\mathbf{k}'$–pairings we associate the map $\mathrm{Kh}'(e) := \mathrm{Kh}(e) \circ (\alpha 1 + \beta X(e))$ where 1 denotes the identity morphism of $\mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}})$ and $X(e)$ is the endomorphism of $\mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}})$ defined below. Given a satisfactory choice of signs, the result is a complex in $\mathrm{Kom}(\mathrm{Mat}(\mathrm{Kob}_{/h}))$, which we denote $[\![D_\mathbf{n}]\!]_{\alpha,\beta}$. We have $[\![D_\mathbf{n}]\!]_{1,0} = [\![D_\mathbf{n}]\!]$. The functors $\mathcal{F}_{\mathrm{Kh}}$ and $\mathcal{F}_{\mathrm{Lee}}$ can be applied to $[\![\cdot]\!]_{\alpha,\beta}$ to obtain homology theories. If β is nonzero, then the topological degree of derivatives is not zero anymore.

The map $X(e)$ is defined as follows. Assume e is an edge between a $\mathbf{k}$–pairing and a $\mathbf{k}'$–pairing, and let C_i and C_{i+1} be the two strands of the cable of L which form a pair in the $\mathbf{k}'$–pairing but not in the $\mathbf{k}$–pairing. Let us choose a point P on C_i which is not a crossing of $D^{\mathbf{n}-2\mathbf{k}}$. Let G be the region of $D^{\mathbf{n}-2\mathbf{k}}$ which lies next to P and between the two components C_i and C_{i+1}. Color the regions of $D^{\mathbf{n}-2\mathbf{k}}$ in a chessboard fashion, such that the unbounded region is colored white, and put $\sigma(G) := +1$ if G is black and $\sigma(G) := -1$ if G is white. Define a cobordism $H(P)$ from $D^{\mathbf{n}-2\mathbf{k}}$ to itself as follows: $H(P)$ is the identity cobordism outside a small neighborhood of P, and it is a composition of two saddle moves near P. The first saddle splits off a small circle from C_i. The second saddle merges the small circle in C_i again. Define $X(e) := (\sigma(G)/2)\,\mathrm{Kh}(H(P))$, where $\mathrm{Kh} : \mathcal{C}ob^4 \to \mathrm{Kob}_{/h}$ is the functor discussed in Section 1.3. We claim that $X(e)$ is independent of the choice of the point P on C_i. Indeed, moving the point P past a crossing of $D^{\mathbf{n}-2\mathbf{k}}$ changes the sign of both $\sigma(G)$ and $\mathrm{Kh}(H(P))$. It is easy to see that $\mathcal{F}_{\mathrm{Lee}}(H(P)) = -\mathcal{F}_{\mathrm{Lee}}(H(P'))$ if P' is obtained from P by moving past a crossings. Moreover, the colors of regions next to P and P' are different. If C_i belongs to the cable of a component K of L, we also use the notation $X(K, i)$ for $X(e)$.

2.5. **Proof of Theorem 1.** Let $D_\mathbf{n}$ and $D'_\mathbf{n}$ be two diagrams representing isotopic colored framed links. Then $D^{\mathbf{n}-2\mathbf{k}}$ and $D'^{\mathbf{n}-2\mathbf{k}}$ represent isotopic links, and hence by Theorem 1.1 the complexes $\mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}})$ and $\mathrm{Kh}(D'^{\mathbf{n}-2\mathbf{k}})$ are isomorphic as objects of $\mathrm{Obj}(\mathrm{Kob}_{/h})$. The isotopy between the links represented by $D^{\mathbf{n}-2\mathbf{k}}$ and $D'^{\mathbf{n}-2\mathbf{k}}$ extends to an isotopy between the annuli appearing in the defintion of the differentials of $[\![D_{\mathbf{n}}]\!]$ and $[\![D'_{\mathbf{n}}]\!]$. Using Theorem 1.2 and Lemma 2.1, it easily follows that $[\![D_{\mathbf{n}}]\!]$ and $[\![D'_{\mathbf{n}}]\!]$ are isomorphic. □

2.6. **Proof of Corollary 2.** The total graded Euler characteristic of $\mathcal{F}_{\mathrm{Kh}}([\![D_{\mathbf{n}}]\!])$ is

$$\begin{aligned}\chi(\mathcal{F}_{\mathrm{Kh}}([\![D_{\mathbf{n}}]\!])) &= \sum_{i,r,j}(-1)^{i+r}q^j \dim_{\mathbb{Q}}(M^{i,r,j}(D_{\mathbf{n}}) \otimes_{Z_{(2)}} \mathbb{Q}) \\ &= \sum_i (-1)^i \sum_{|\mathbf{k}|=i}\sum_{\mathbf{s}\in\mathbf{k}} \chi(\mathcal{F}_{\mathrm{Kh}}(\mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}}))) \\ &= \sum_{\mathbf{k}=\mathbf{0}}^{\lfloor \mathbf{n}/2 \rfloor} (-1)^{|\mathbf{k}|} \begin{pmatrix} \mathbf{n}-\mathbf{k} \\ \mathbf{k} \end{pmatrix} \chi(\mathcal{F}_{\mathrm{Kh}}(\mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}}))) \,.\end{aligned}$$

Taking into account that $\chi(\mathcal{F}_{\mathrm{Kh}}(\mathrm{Kh}(D^{\mathbf{n}-2\mathbf{k}}))) = J(L^{\mathbf{n}-2\mathbf{k}})$ we get the result. □

3. Framed cobordisms

3.1. **Framings for submanifolds of codimension 2.** Let M be a smooth oriented n–manifold and $N \subset M$ a compact smooth oriented submanifold of M. By a framing of N we mean a trivialization of its normal bundle ν_N in M. Note that a smooth ambient isotopy between submanifolds induces an isomorphism between their normal bundles. Hence it makes sense to compare framings of ambient isotopic submanifolds. Given a trivialization $f : \nu_N|_{\partial N} \to \partial N \times \mathbb{R}^2$, we define a relative framing of N, relative to f, as a trivialization of ν_N which restricts to f on ∂N. Relative isomorphism classes of oriented 2–plane bundles over N which are trivialized over ∂N, correspond to homotopy classes of maps from $(N, \partial N)$ to $(BSO(2), p_0)$, where p_0 is an arbitrary basepoint in $BSO(2)$. Since $BSO(2)$ is a $K(\mathbb{Z}, 2)$ space, we have $[N, \partial N; BSO(2), p_0] = H^2(N, \partial N) = H_{n-4}(N)$. N admits a relative framing if and only if (ν_N, f) corresponds to the zero class in $H_{n-4}(N)$. In that case, the set of all relative framings is an affine space over $[N, \partial N; SO(2), 1] = H^1(N, \partial N) = H_{n-3}(N)$.

We are mainly interested in the case where N is connected and $n = 4$. In this case the obstruction for the existence of relative framings is an integer $e(\nu_N, f) \in H_0(N) = \mathbb{Z}$ which we call the relative Euler number of ν_N. The relative Euler number can be described explicitly as follows: let s be the zero section of ν_N and s' a generic section such that $f(s'(x)) = (x, e_1)$ for $x \in \partial N$ where e_1 denotes the first basis vector of $\mathbb{R}^2$. Then $e(\nu_N, f) = s \cdot s'$ where $s \cdot s'$ denotes the algebraic intersection number of the surfaces s and s' in the total space of ν_N. N has a tubular neighborhood in M which is diffeomorphic to the total space of ν_N. Therefore, the relative Euler number $e(\nu_N, f)$ can be computed as a "relative self–intersection number" of N in M.

3.2. **Framings for links and link cobordisms.** Let $K = N$ be a knot in $\mathbb{R}^3$. We can specify a framing of K by a vector field on K which is nowhere tangent to K. If the vectors are sufficiently short, their tips trace out a knot K' parallel to K. Recall that the framing coefficient $n(f)$ is defined as the linking number of K and K'.

Let us give an alternative description of the framing coefficient. Let $S \subset \mathbb{R}^3 \times I$ be a connected cobordism between the empty link and the framed knot K, i.e. $\partial S = K \subset \mathbb{R}^3 \times \{1\}$. We assume that S is parallel to the I direction in a neighborhood of ∂S, such that the restriction $\nu_S|_{\partial S}$ coincides with the normal bundle of ∂S in $\mathbb{R}^3 \times \{1\}$. Then it makes sense to consider the relative Euler number $e(\nu_S, f)$ where f is the framing of K. We claim that $e(\nu_S, f) = n(f)$. We only prove that $e(\nu_S, f)$ is independent of the choice of S: let S, S_1 be two cobordisms from the empty link to K and let $\bar{S}_1$ denote the cobordism S_1 "turned upside down". The composition of S and $\bar{S}_1$ is a closed surface $F := S \cup \bar{S}_1$. Consider small perturbations S' and S_1' of S and S_1 with $\partial S' = \partial S_1' = K'$ and let $F' := S' \cup \bar{S}_1'$. We have $e(\nu_S, f) - e(\nu_{S_1}, f) = S \cdot S' + \bar{S}_1 \cdot \bar{S}_1' = F \cdot F' = 0$, where we have used that F has self–intersection number zero because $H_2(\mathbb{R}^3 \times I) = 0$. Hence $e(\nu_S, f)$ is independent of S.

Now let S be a cobordism connecting two framed knots (K_0, f_0) and (K_1, f_1). Choose cobordisms S_0 and S_1 from the empty link to K_0 and K_1, respectively. By considering small perturbations S', S_0', S_1' as above, we obtain $0 = S_0 \cdot S_0' + S \cdot S' + \bar{S}_1 \cdot \bar{S}_1' = n(f_0) + e(\nu_S, f_0 \cup f_1) - n(f_1)$. Hence S admits a relative framing if and only if $e(\nu_S, f_0 \cup f_1) = 0$ if and only if $n(f_0) = n(f_1)$.

Let us also consider the case of framed links. If L is a link of $|L|$ components in $\mathbb{R}^3$, a framing of L can be described by an $|L|$–tuple $(n(f_1), \cdots, n(f_{|L|})) \in \mathbb{Z}^{|L|}$ where f_i denotes the restriction of the framing to the ith component. We define the total framing coefficient as

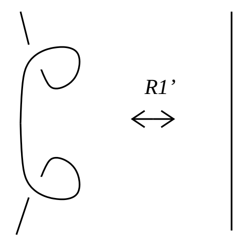
$$n(f) := n(f_1) + \ldots + n(f_{|L|}) + \sum_{i \neq j} lk(L_i, L_j).$$

It is easy to see that $n(f) = e(\nu_S, f)$ for any connected cobordism S from the empty link to L. Arguing as above, we conclude that two framed links may be connected by a relatively framed cobordism if and only if their total framing coefficients agree.

If the set of relative framings of S is non–empty, it is an affine space over $H_1(S)$. The action of $H_1(S)$ can be seen as follows: let c be an oriented simple closed curve on S representing an element of $H_1(S)$. Consider a tubular neighborhood U of c, diffeomorphic to $c \times [0, 2\pi]$. Let χ_c be the map from S to $SO(2)$ which is trivial on the complement of U and maps a point $(\theta, \varphi) \in U = c \times [0, 2\pi]$ to rotation by φ. Then c acts on framings by sending the framing given by a vector field $v(z)$ to the framing given by the vector field $\chi_c(z)v(z)$. In this context, the

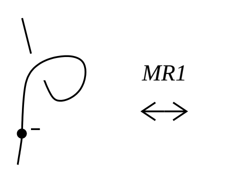
Poincaré dual $\mathrm{PD}^{-1}[c] \in H^1(S, \partial S)$ has the following interpretation: let c' be a properly embedded simple curve on S representing an element of $H_1(S, \partial S)$. The restriction $\chi_c|_{c'}$ is a closed curve in $SO(2)$ whose class in $\pi_1(SO(2), 1) = \mathbb{Z}$ is given by $[\chi_c|_{c'}] = c \cdot c' = \langle \mathrm{PD}^{-1}[c], [c'] \rangle$.

3.3. **Link diagrams with marked points.** Let L be a link and D a diagram of L. We may use D to specify a framing on L, namely the framing given by vector field on L which is everywhere perpendicular to the plane of D. This framing is called the blackboard framing. It allows us to view link diagrams as diagrams of framed links. The blackboard framing is invariant under the second and the third Reidemeister moves, but not under the first Reidemeister move. It is easy to see that two link diagrams describe isotopic framed links if and only the differ by a sequence of the following moves: the modified first Reidemeister move R1' shown below,



as well as the second and the third Reidemeister move.

A link diagram with marked points is a link diagram D together with a finite collection of distinct points, lying on the interiors of the edges of D, and labeled with $+$ or $-$. If D is a link diagram with marked points, the writhe $wr(D)$ is the difference between the numbers of positive and negative crossings in D. The twist $tw(D)$ is the difference between the numbers of positive and negative marked points. A link diagram with marked points determines a framing f_D of L as follows: f_D is given by a vector field which is perpendicular to the drawing plane, except in a small neighborhood of the marked points, where it twists around the link, such that each positive point contributes $+1$ to $n(f_D)$ and each negative point contributes -1 to $n(f_D)$. Thus we have $n(f_D) = F \cdot L' = wr(D) + tw(D)$.



The marked first Reidemeister move MR1, shown above, leaves $n(f_D)$ unchanged. It follows that two diagrams with marked points describe isotopic framed links if and only if they are related by a finite sequence of the following moves: marked first

Reidemeister move MR1, Reidemeister moves R2 and R3, creation/annihilation of a pair of nearby oppositely marked points and sliding a marked point past a crossing.

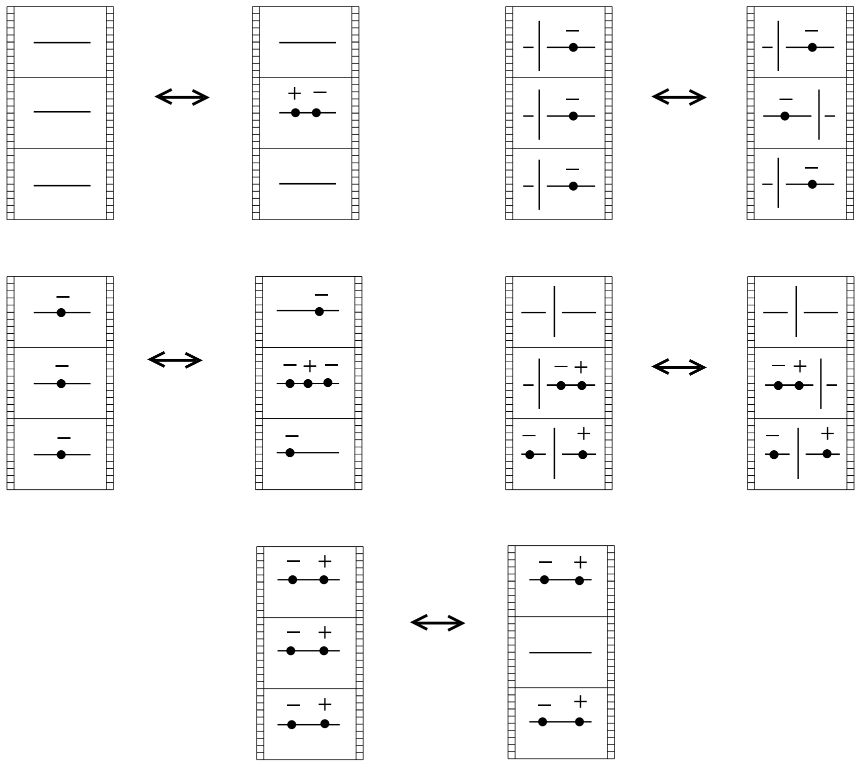


Figure 4 *Additional movie moves for framed cobordisms*

3.4. **Movie presentations for framed cobordisms.** In this subsection, we discuss movie presentations for framed link cobordisms.

Let S be an unframed cobordism, presented as a sequence of link diagrams. If there are two consecutive link diagrams differing by an R1 move, we introduce marked points in the movie presentation, such that every R1 move becomes an MR1 move. The result is a movie of link diagrams with marked points, describing a movie of framed links. The framings of these links determine a well-defined framing of S. We claim that every framing of S arises in this way. To prove this claim, it would be sufficient to check that it is true for elementary cobordisms (caps, cups and saddles). However, we give a different proof. The marked points in the movie presentation trace out curves on the cobordism S (note that these curves may have local extrema corresponding to annihilation and creation of marked points). We can orient these curves consistently, by declaring that positive points "move" in negative I direction and negative points move in positive I direction. Conversely, if c is an oriented simple closed curve on S, we can think of c as consisting of lines traced out by marked points. We can insert these marked points into a given movie presentation of S. Thus, c acts on movie presentations of S by insertion of marked points. This action induces an action on the framings of S which are described by the movie presentations. It is easy to see that the action on the framings coincides with the action of $H_1(S)$ discussed at the end of Subsection 3.2. Since $H_1(S)$ acts transitively, it follows that every framing of S can be described by a movie presentation. Moreover, two oriented simple closed curves induce equivalent actions on framings if and only if they are homologous.

Proof of Theorem 3. The local movie moves which are sufficient to relate any two homologous curves traced out by marked points on a cobordism are shown in Figure 4. These movie moves, together with modifications of the Carter-Saito movie moves obtained by inserting marked points, are sufficient to relate any two movie presentations of isotopic framed cobordisms. □

We can transform a link diagram with marked points into a link diagram without marked points by inserting a left–twist curl for each point marked with a + and a right–twist curl for each point marked with a −. Under this substitution, sliding a marked point past a crossing becomes a composition of the second and the third Reidemeister move. The MR1 move and creation/annihilation of a pair of oppositely marked points become the modified first Reidemeister move R1'.

We can transform a movie presentation without marked points into a movie presentation with marked points as follows: we replace each R1' move by a composition of two opposite MR1 moves. The two opposite MR1 moves create a pair of oppositely marked points. We annihilate this pair immediately after its creation. The resulting movie presentation with marked points differs from the original movie presentation only locally. Since we already know movie moves for movie presentations with marked points, we can define movie moves for movie presentations without marked points simply by replacing marked points with curls.

4. Colored framed cobordisms

Let $\mathcal{C}\mathbf{ob}^4_f$ be the category of colored framed movie presentations. The objects are diagrams of colored links and the morphisms movie presentations of colored framed links, i.e. sequences of colored framed link diagrams, where between two consecutive diagrams one of the following transformations occur — R1', R2 or R3 move, a saddle, a cap or a cup. Note that here we need to distinguish between two saddle moves: a "splitting" saddle which splits one colored component into two of the same color, and a "merging" saddle which merges two components of *the same* *color* into one component. To components colored differently the merging saddle can not be applied.

We are interested in a construction of a functor $\mathcal{K}\mathbf{h}_{\alpha,\beta} : \mathcal{C}\mathbf{ob}^4_f \to \mathrm{Kom}(\mathrm{Mat}(\mathrm{Kob}_h))$. For the objects we put $\mathcal{K}\mathbf{h}_{\alpha,\beta}(D_\mathbf{n}) = [\![D_\mathbf{n}]\!]_{\alpha,\beta}$. The rest of this section is devoted to the definition of chain transformations corresponding to cap and cup, and saddles. The Reidemeister moves induce chain homotopies defined in [3].

We introduce the following notation. Let $D_\mathbf{n}$ be a colored link diagram, and let $\mathbf{s}$ be a $\mathbf{k}$–pairing of the $\mathbf{n}$–cable of D. Then we set $D^\mathbf{s} := D^{\mathbf{n}-2\mathbf{k}}$, where the strands of $D^{\mathbf{n}-2\mathbf{k}}$ correspond to the dots which are not contained in a pair of $\mathbf{s}$.

4.1. **Cup and cap.** Consider two diagrams D and D_0 which are related by a cap cobordism. Assume that D_0 is the disjoint union of D with a trivial component K. Let $\mathbf{n}_0$ be a coloring of D_0 and let $\mathbf{n}$ denote the induced coloring of D. Let n denote the restriction of $\mathbf{n}_0$ to K. Let $\mathbf{s}$ be a pairing of the $\mathbf{n}$–cable of D and let $\mathbf{s}_0$ be a pairing of the $\mathbf{n}_0$–cable of D_0. We define a morphism $\iota^{\mathbf{s}_0,\mathbf{s}} : \mathrm{Kh}(D^\mathbf{s}) \to \mathrm{Kh}(D_0^{\mathbf{s}_0})$ as follows: $\iota^{\mathbf{s}_0,\mathbf{s}}$ is nonzero only if the restriction of $\mathbf{s}_0$ to K is the empty pairing (no pairs) and if $\mathbf{s}_0$ agrees with $\mathbf{s}$ on all other components. In this case, we define $\iota^{\mathbf{s}_0,\mathbf{s}}$ as the composition of the following two morphisms: the morphism induced by a union of n caps whose boundaries are the n strands of the n–cable of K, and the endomorphism φ of $\mathrm{Kh}(D_0^{\mathbf{s}_0})$ given by $\varphi := \sum_{j=1}^n A_j \circ B_j$, where A_j denotes the composition of all morphisms $(\alpha 1 - \beta X(K,i))/2$ for $1 \le i \le j$, and B_j denotes the composition of all morphisms $(\alpha 1 + \beta X(K,i))/2$ for $j < i \le n$.

Now let D and D_0 be two link diagrams related by a cup cobordism. Assume that D is the disjoint union of D_0 with a trivial component K. Let $\mathbf{n}$ be a coloring of D and let $\mathbf{n}_0$ denote the induced coloring of D_0. Let n denote the restriction of $\mathbf{n}$ to K. Let $\mathbf{s}$ be a pairing of the $\mathbf{n}$–cable of D and let $\mathbf{s}_0$ be a pairing of the $\mathbf{n}_0$–cable of D_0. We define a morphism $\epsilon^{\mathbf{s}_0,\mathbf{s}} : \mathrm{Kh}(D^\mathbf{s}) \to \mathrm{Kh}(D_0^{\mathbf{s}_0})$ as follows: $\epsilon^{\mathbf{s}_0,\mathbf{s}}$ is nonzero only if the restriction of $\mathbf{s}$ to K is the empty pairing and if $\mathbf{s}$ agrees with $\mathbf{s}_0$ on all other components. In this case, we define $\epsilon^{\mathbf{s}_0,\mathbf{s}}$ as the composition of the following two morphisms: the endomorphism φ defined as above and the morphism induced by n cups whose boundaries are the n strands of the n–cable of K.

4.2. **Merging saddle.** Consider two diagrams D and D_0 which are related by a saddle merging two components K_1 and K_2 of D into a single component K of D_0. Let $\mathbf{n}$ be a coloring of D, such that K_1 and K_2 have the same color n. Let $\mathbf{n}_0$ be the induced coloring of D_0. Consider a pairing $\mathbf{s}$ of the $\mathbf{n}$–cable of D, and let s_1 and s_2 denote the restrictions of $\mathbf{s}$ to K_1 and K_2, respectively. Let $s_1 s_2$ denote the union of s_1 and s_2, i.e. the pairing which consists of all pairs which are contained in either s_1 or s_2.

Let $\gamma, \delta \in \mathbb{Z}$. Given a pairing $\mathbf{s}_0$ of the $\mathbf{n}_0$–cable of D_0, the morphism

$$\psi^{\mathbf{s}_0,\mathbf{s}}_{\gamma,\delta} : \mathrm{Kh}(D^{\mathbf{s}}) \longrightarrow \mathrm{Kh}(D^{\mathbf{s}_0})$$

is nonzero only if the following is satisfied:

- s_1 and s_2 have no common dot (meaning that there is no dot which belongs to a pair both in s_1 and in s_2),
- $\mathbf{s}_0$ is the pairing which restricts to $s_1 s_2$ on K and which agrees with $\mathbf{s}$ on all other components.

In this case, we define $\psi^{\mathbf{s}_0,\mathbf{s}}_{\gamma,\delta}$ as follows. For each pair of s_2, consisting of dots numbered i and $i+1$, consider the endomorphism $(\gamma 1 + \delta X(K_1, i))/2$ of $\mathrm{Kh}(D^{\mathbf{s}})$. Similarly, for each pair of s_1, consisting of dots numbered i and $i+1$, consider the endomorphism $(\gamma 1 + \delta X(K_2, i))/2$ of $\mathrm{Kh}(D^{\mathbf{s}})$. Denote the composition of these endomorphisms by ψ_1. Let $\mathbf{s}'$ be the $\mathbf{n}$–pairing which restricts to $s_1 s_2$ on both K_1 and K_2 and which agrees with the $\mathbf{n}$–pairing $\mathbf{s}$ on all other components of D. Define a morphism ψ_2 from $\mathrm{Kh}(D^{\mathbf{s}})$ to $\mathrm{Kh}(D^{\mathbf{s}'})$ as follows: for each pair of s_2, consisting of dots numbered i and $i+1$, consider an annulus attached to the strands numbered i and $i+1$ of K^{s_1}. Similarly, for each pair of s_1, consisting of dots numbered i and $i+1$, consider an annulus attached to the strands numbered i and $i+1$ of K^{s_2}. Let ψ_2 be the morphism induced by these annuli. For every strand of $K_1^{s_1 s_2}$ there is a corresponding strand in $K_2^{s_1 s_2}$. Let ψ_3 be the morphism from $\mathrm{Kh}(D^{\mathbf{s}'})$ to $\mathrm{Kh}(D_0^{\mathbf{s}_0})$ induced by merging each pair of corresponding strands on $K_1^{s_1 s_2}$ and $K_2^{s_1 s_2}$ by a saddle cobordism. Define $\psi^{\mathbf{s}_0,\mathbf{s}}_{\gamma,\delta}$ as the composition $\psi^{\mathbf{s}_0,\mathbf{s}}_{\gamma,\delta} := \psi_3 \psi_2 \psi_1$.

Note that our definition of the morphism $\psi^{\mathbf{s}_0,\mathbf{s}}_{\gamma,\delta}$ mimics the definition of the map ψ in [7]. Khovanov's map ψ corresponds to our morphism $\psi^{\mathbf{s}_0,\mathbf{s}}_{0,2}$, with the difference that we work with the Khovanov bracket whereas Khovanov worked with Khovanov homology over $\mathbb{Z}/2\mathbb{Z}$ coefficients. Note that $\mathcal{F}_{\mathrm{Kh}}(\psi^{\mathbf{s}_0,\mathbf{s}}_{0,\delta})$ is graded of degree $-n$ (where n is the color of the components involved in the saddle move, see above). We denote by $\psi_{\gamma,\delta}$ the collection of all morphisms $\psi^{\mathbf{s}_0,\mathbf{s}}_{\gamma,\delta}$.

4.3. **Splitting saddle.** Suppose the diagrams D and D_0 are related by a saddle which splits a component K of D into two components K_1 and K_2 of D_0. Let $\mathbf{n}$ be a coloring of D, and let $\mathbf{n}_0$ be the induced coloring of D_0. Consider a pairing $\mathbf{s}$ of the $\mathbf{n}$–cable of D which restricts to a k–pairing s on K.

Let $\gamma, \delta \in \mathbb{Z}$. Given a pairing $\mathbf{s}_0$ of the $\mathbf{n}_0$–cable of D_0, the morphism

$$\bar{\psi}^{\mathbf{s}_0,\mathbf{s}}_{\gamma,\delta} : \mathrm{Kh}(D^{\mathbf{s}}) \longrightarrow \mathrm{Kh}(D_0^{\mathbf{s}_0})$$

is zero unless $\mathbf{s}_0$ has the following properties:

- the restrictions s_1 and s_2 of $\mathbf{s}_0$ to K_1 and K_2 have no common dot,

- the union of s_1 and s_2 is equal to s,
- $\mathbf{s}_0$ agrees with $\mathbf{s}$ on all components of D_0 other than K_1 and K_2.

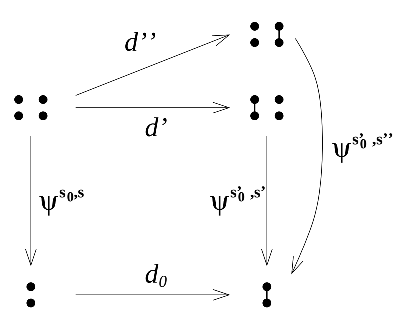
For an $\mathbf{s}_0$ with these properties, we define $\bar{\psi}^{\mathbf{s}_0,\mathbf{s}}_{\gamma,\delta} := 2^k\bar{\psi}_1\bar{\psi}_2\bar{\psi}_3$, where $\bar{\psi}_1, \bar{\psi}_2$ and $\bar{\psi}_3$ are the morphisms obtained by turning the morphisms ψ_1, ψ_2 and ψ_3 of Subsection 4.2 upside down.

Note that $\bar{\psi}^{\mathbf{s}_0,\mathbf{s}}_{0,\delta}$ is graded of degree $-n$, where n is the color of the components involved in the saddle move. We denote by $\bar{\psi}_{\gamma,\delta}$ the collection of all morphisms $\bar{\psi}^{\mathbf{s}_0,\mathbf{s}}_{\gamma,\delta}$.

4.4. **Chain transformations induced by saddles.** Consider two diagrams D and D_0 which are related by a merging saddle. Assume we are given a collection of morphisms $\psi = \{\psi^{\mathbf{s}_0,\mathbf{s}}\}$ as in Subsection 4.2 (we drop the subscripts γ, δ to simplify the notation). We wish to have a criterion under which ψ induces a chain transformation from $[\![D_{\mathbf{n}}]\!]_{\alpha,\beta}$ to $[\![D_{0,\mathbf{n}_0}]\!]_{\alpha,\beta}$.

Let d and d_0 denote the differentials of $[\![D_{\mathbf{n}}]\!]_{\alpha,\beta}$ and $[\![D_{0,\mathbf{n}_0}]\!]_{\alpha,\beta}$, respectively. Both $d_0\psi$ and ψd increase the height (the homological degree) by one. Let $\mathbf{s}$ be a pairing of the $\mathbf{n}$–cable of D, and let $\mathbf{s}_0'$ be a pairing of the $\mathbf{n}_0$–cable of D_0 whose height is one larger than the height of $\mathbf{s}$. Let $(d_0\psi)^{\mathbf{s}_0',\mathbf{s}}$ denote the "restriction" of $d_0\psi$ to $\mathrm{Kh}(D^{\mathbf{s}})$ and $\mathrm{Kh}(D_0^{\mathbf{s}_0'})$, and let $(\psi d)^{\mathbf{s}_0',\mathbf{s}}$ denote the "restriction" of ψd to $\mathrm{Kh}(D^{\mathbf{s}})$ and $\mathrm{Kh}(D_0^{\mathbf{s}_0'})$. Assume that at least one of the morphisms $(d_0\psi)^{\mathbf{s}_0',\mathbf{s}}$ and $(\psi d)^{\mathbf{s}_0',\mathbf{s}}$ is nonzero. This is only possible if the restrictions of $\mathbf{s}$ to the two components involved in the saddle move have no common dot. Moreover, all pairs of the pairing $\mathbf{s}_0$ (defined as in Subsection 4.2) must also be pairs of $\mathbf{s}_0'$. Therefore, $\mathbf{s}_0'$ must contain a unique pair π which is not contained in $\mathbf{s}_0$. We assume that π belongs to the component of D_0 which is involved in the saddle move (for otherwise $(d_0\psi)^{\mathbf{s}_0',\mathbf{s}} = \pm(\psi d)^{\mathbf{s}_0',\mathbf{s}}$ is trivially satisfied). Then we are in the situation of (3), where the pair in the lower right corner is the pair π, and where we have left away all dots corresponding to strands on which $(d_0\psi)^{\mathbf{s}_0',\mathbf{s}}$ and $(\psi d)^{\mathbf{s}_0',\mathbf{s}}$ agree already by definition.

(3)

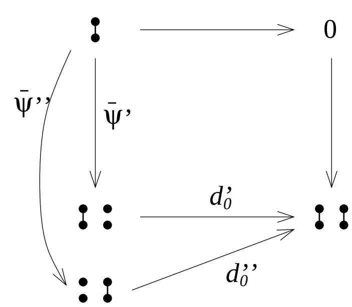


Lemma 4.1. *Assume that* $d_0\psi^{\mathbf{s}_0,\mathbf{s}} = \pm(\psi^{\mathbf{s}_0',\mathbf{s}'}d' + \psi^{\mathbf{s}_0',\mathbf{s}''}d'')$ *for all squares as in (3). Then there is a* 0*–cochain* $\gamma \in C^0(\Gamma_{\mathbf{n}}, \mathbb{Z}/2\mathbb{Z})$ *such that the morphisms* $(-1)^{\gamma(\mathbf{s})}\psi^{\mathbf{s}_0,\mathbf{s}}$ *determine a chain transformation from* $[\![D_{\mathbf{n}}]\!]_{\alpha,\beta}$ *to* $[\![D_{0,\mathbf{n}_0}]\!]_{\alpha,\beta}$.

Proof. Consider the subgraph $\Gamma_{\mathbf{n}}'$ of $\Gamma_{\mathbf{n}}$ whose vertices are precisely those $\mathbf{n}$–pairings $\mathbf{s}$ whose restrictions to the two components involved in the saddle move have no common dot. Let $f : \Gamma_{\mathbf{n}}' \to \Gamma_{\mathbf{n}_0}$ be the map which maps a pairing/vertex $\mathbf{s}$ to the induced pairing/vertex $\mathbf{s}_0$. Note that every edge e_0 of $\Gamma_{\mathbf{n}_0}$ appears as the lower edge of a square as in (3). Define a 1–cochain $\zeta \in C^1(\Gamma_{\mathbf{n}_0}, \mathbb{Z}/2\mathbb{Z})$ by setting $\zeta(e_0) := 0$ if $d_0\psi^{\mathbf{s}_0,\mathbf{s}} = +(\psi^{\mathbf{s}_0',\mathbf{s}'}d' + \psi^{\mathbf{s}_0',\mathbf{s}''}d'')$ and $\zeta(e_0) := 1$ if $d_0\psi^{\mathbf{s}_0,\mathbf{s}} = -(\psi^{\mathbf{s}_0',\mathbf{s}'}d' + \psi^{\mathbf{s}_0',\mathbf{s}''}d'')$. Since f maps squares of $\Gamma_{\mathbf{n}}'$ to squares of $\Gamma_{\mathbf{n}_0}$, and since all squares of $\Gamma_{\mathbf{n}}'$ and $\Gamma_{\mathbf{n}_0}$ anticommute, the 1–cochain $f^*\zeta \in C^1(\Gamma_{\mathbf{n}}', \mathbb{Z}/2\mathbb{Z})$ maps all squares of $\Gamma_{\mathbf{n}}'$ to zero. Hence $f^*\zeta = \delta\gamma'$ for a 0–cochain $\gamma' \in C^0(\Gamma_{\mathbf{n}}', \mathbb{Z}/2\mathbb{Z})$. Now the 0–cochain $\gamma \in C^0(\Gamma_{\mathbf{n}}, \mathbb{Z}/2\mathbb{Z})$ in the statement of the lemma is an arbitrary extension of γ'. □

Now assume D and D_0 are related by a splitting saddle and assume we are given a collection of morphisms $\bar\psi = \{\bar\psi^{\mathbf{s}_0,\mathbf{s}}\}$ as in Subsection 4.3. Let $\mathbf{s}$ be a pairing of the $\mathbf{n}$–cable of D and let $\mathbf{s}_0'$ be a pairing of the $\mathbf{n}_0$–cable of D_0, such that at least one of the morphisms $(d_0\bar\psi)^{\mathbf{s}_0',\mathbf{s}}$ and $(\bar\psi d)^{\mathbf{s}_0',\mathbf{s}}$ is nonzero. Let K be the component of D which is involved in the saddle and let s be the restriction of $\mathbf{s}$ to K. Similarly, let K_1 and K_2 be the components of D_0 which are involved in the saddle and let s_1' and s_2' denote the restrictions of $\mathbf{s}_0'$ to K_1 and K_2. Then every pair of s must also appear in the union $s_1' \cup s_2'$ (where we regard s_1' and s_2' as sets of pairs). If s_1' and s_2' have a common pair, we are in the situation of (4), where $\mathbf{s}_0'$ is the pairing in the lower right corner.

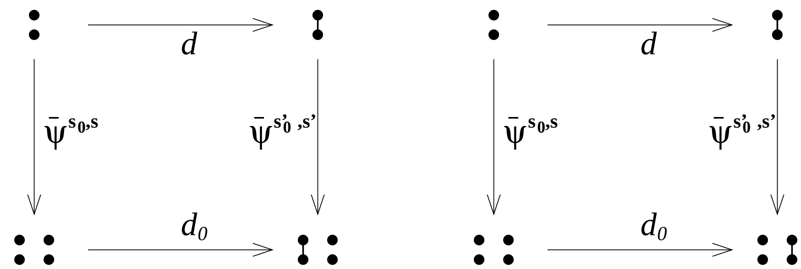
(4)



Now assume that s_1' and s_2' have no common pair. Let s_1 and s_2 denote the intersections $s_1 := s \cap s_1'$ and $s_2 := s \cap s_2'$. Let $\mathbf{s}_0$ denote the pairing of the $\mathbf{n}_0$–cable of D_0 which restricts to s_1 and s_2 on the components K_1 and K_2 and which agrees with $\mathbf{s}$ on all other components of D_0. Then every pair of $\mathbf{s}_0$ is also be a pair of $\mathbf{s}_0'$, and there is a unique pair π of $\mathbf{s}_0'$ which is not contained in $\mathbf{s}_0$. We assume that π

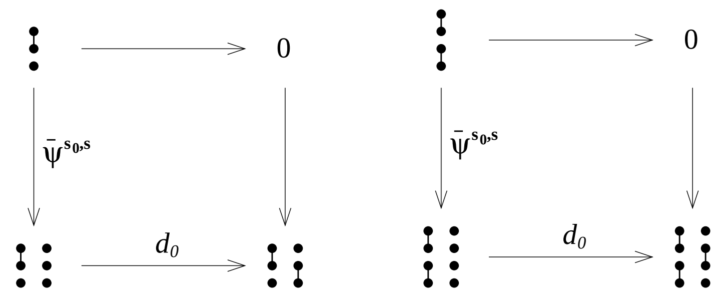
belongs K_1 or K_2 (for otherwise $(d_0\bar{\psi})^{\mathbf{s}_0',\mathbf{s}} = \pm(\bar{\psi}d)^{\mathbf{s}_0',\mathbf{s}}$ is trivially satisfied). If π is disjoint from all pairs of $s_1 \cup s_2$, we are in the situation of (5), where π is the pair in the lower right corner.

(5)



It is also possible that π has a common dot with a pair of $s_1 \cup s_2$. Examples of this case are shown in (6).

(6)



Lemma 4.2. *Assume that the squares of (5) commute, up to sign, and assume* $d_0\bar{\psi}^{\mathbf{s}_0,\mathbf{s}} = 0$ *for all squares as in (6). Then there is a 0–cochain* $\gamma \in C^0(\Gamma_{\mathbf{n}_0}, \mathbb{Z}/2\mathbb{Z})$ *such that the morphisms* $(-1)^{\gamma(\mathbf{s}_0)}\bar{\psi}^{\mathbf{s}_0,\mathbf{s}}$ *determine a chain transformation from* $[\![D_{\mathbf{n}}]\!]_{\alpha,\beta}$ *to* $[\![D_{0,\mathbf{n}_0}]\!]_{\alpha,\beta}$.

Proof. The proof is analogous to the proof of Lemma 4.1 (although now we have to consider a map f going from a subgraph $\Gamma'_{\mathbf{n}_0}$ of $\Gamma_{\mathbf{n}_0}$ to $\Gamma_{\mathbf{n}}$). Note that the morphisms $d_0'\bar{\psi}'$ and $d_0''\bar{\psi}'$ of (4) cancel automatically if the squares of (5) commute. To see this, observe that the morphisms $\bar{\psi}'$ and $\bar{\psi}''$ of (4) also appear in the squares of (5). Moreover, the lower edges of the squares in (4) and (5) form a square of $\Gamma_{\mathbf{n}_0}$. Now use that the squares of $\Gamma_{\mathbf{n}_0}$ anticommute. □

5. Towards functoriality

Throughout this section let $\mathcal{A}$ be the category of $\mathbb{Z}_{(2)}$–modules. Recall that $\mathcal{F}_{\mathrm{Lee}}, \mathcal{F}_{\mathrm{Kh}} : \mathcal{C}ob^3_{/l} \to \mathcal{A}$ extend to the functors $\mathcal{F}_{\mathrm{Lee}}, \mathcal{F}_{\mathrm{Kh}} : \mathrm{Kom}(\mathrm{Mat}(\mathrm{Kob}_{/h})) \to$

$\mathrm{Kom}(\mathrm{Kom}_{/h}(\mathcal{A}))$. The isomorphism classes of $\mathcal{F}_{\mathrm{Lee}}([\![D_n]\!]_{\alpha,\beta})$ and $\mathcal{F}_{\mathrm{Kh}}([\![D_n]\!]_{\alpha,\beta})$ are invariants of a colored link.

Theorem 5.1. *For $\alpha = \beta = 1$, the maps $\mathcal{F}_{\mathrm{Lee}}(\psi_{1,1}^{\mathbf{s}_0,\mathbf{s}})$ and $\mathcal{F}_{\mathrm{Lee}}(\bar{\psi}_{1,1}^{\mathbf{s}_0,\mathbf{s}})$ induce chain transformations.*

Before we prove the theorem, let us introduce a new relation in $\mathcal{C}ob^3$, called the genus reduction relation. Consider a cobordism C' obtained from a cobordism C by attaching two small handles to a disk of C. The genus reduction relation asserts that $C' = 4C$. Now let us assume that 2 is invertible and that the relations (S), (T) and (4Tu) hold. Then the genus reduction relation becomes equivalent to the relation  $= 8$, i.e. to the defining relation for Lee's functor. As a consequence, $\mathcal{F}_{\mathrm{Lee}}(\mathrm{Kh}(H(P)))^2/4$ is the identity map, for $H(P)$ defined as in Subsection 2.4.

Proof of Theorem 5.1. Assume $\alpha = \beta = 1$ and assume that the genus reduction relation holds. We have to show that under this assumption, the morphisms $\psi_{1,1}^{\mathbf{s}_0,\mathbf{s}}$ and $\bar{\psi}_{1,1}^{\mathbf{s}_0,\mathbf{s}}$ satisfy the conditions of Lemmas 4.1 and 4.2, respectively. We start by proving $d_0\bar{\psi}_{1,1}^{\mathbf{s}_0,\mathbf{s}} = 0$ for the left square of (6) (the proof for the right square is analogous). We assume that the three dots in the lower left corner of the square are numbered from bottom to top from i to $i+2$. Consider the diagram $D_0^{\mathbf{n}_0}$ of the $\mathbf{n}_0$–cable of D_0. Let l_i, l_{i+1}, and l_{i+2} be three parallel edges of $D_0^{\mathbf{n}_0}$, belonging to the strands C_i, C_{i+1} and C_{i+2}, respectively. Let G_i denote region of $D_0^{\mathbf{n}_0}$ which lies between l_i and l_{i+1}, and let G_{i+1} denote the region which lies between l_{i+1} and l_{i+2}. Choose a point P_i on l_i and a point P_{i+1} on l_{i+1}. Observe that on C_{i+2}, the map $\bar{\psi}_{1,1}^{\mathbf{s}_0,\mathbf{s}}$ is induced by a saddle cobordism. On C_i and C_{i+1}, it is induced by an annulus postcomposed with $(1 + (\sigma(G_i)/2)\,\mathrm{Kh}(H(P_i)))$. We can replace $\mathrm{Kh}(H(P_i))$ by $\mathrm{Kh}(H(P_{i+1}))$ because we can move the point P_i across the annulus. For $\alpha = \beta = 1$, the map d_0 is given by $(1 + (\sigma(G_{i+1})/2)\,\mathrm{Kh}(H(P_{i+1})))$, postcomposed with an annulus. Since G_i and G_{i+1} are neighbors, $\sigma(G_{i+1}) = -\sigma(G_i)$. Summarizing, we see that $d_0\bar{\psi}_{1,1}^{\mathbf{s}_0,\mathbf{s}}$ factors through $(1-(\sigma(G_i)/2)\,\mathrm{Kh}(H(P_{i+1})))(1+(\sigma(G_i)/2)\,\mathrm{Kh}(H(P_{i+1}))) = 1-\mathrm{Kh}(H(P_{i+1}))^2/4 = 0$.

To show that the squares of (3) and (5) commute (up to sign), apply isotopies, the neck cutting relation and the genus reduction relation to the cobordisms corresponding to $(d_0\psi_{1,1})^{\mathbf{s}_0',\mathbf{s}}$, $(\psi_{1,1}d)^{\mathbf{s}_0',\mathbf{s}}$, $(d_0\bar{\psi}_{1,1})^{\mathbf{s}_0',\mathbf{s}}$ and $(\bar{\psi}_{1,1}d)^{\mathbf{s}_0',\mathbf{s}}$. Use that $(\sigma(G)/2)\,\mathrm{Kh}(H(P))$ is independent of the choice of the point P on C_i, and that $\mathrm{Kh}(H(P))$ commutes with morphisms induced by cobordisms which agree with the identity cobordism in a neighborhood of P. The details are left to the reader. □

Theorem 5.2. *For $\alpha = 0$, $\beta = 1$, the maps $\mathcal{F}_{\mathrm{Kh}}(\psi_{0,1}^{\mathbf{s}_0,\mathbf{s}})$ and $\mathcal{F}_{\mathrm{Kh}}(\bar{\psi}_{0,1}^{\mathbf{s}_0,\mathbf{s}})$ induce chain transformations.*

Proof. Let us first show that $\mathcal{F}_{\mathrm{Kh}}(d_0\bar{\psi}^{\mathbf{s}_0,\mathbf{s}}_{0,1}) = 0$ for the left square of (6) (the proof for the right square is analogous). By the same arguments as in the previous proof, $\mathcal{F}_{\mathrm{Kh}}(d_0\bar{\psi}^{\mathbf{s}_0,\mathbf{s}}_{1,1})$ factors through $\mathcal{F}_{\mathrm{Kh}}(\mathrm{Kh}(H(P_{i+1}))\,\mathrm{Kh}(H(P_{i+1}))) = 0$, because the genus of the composition is bigger than one.

To show that the squares of (3) and (5) commute (up to sign), we have to proceed like in the previous proof, replacing the genus reduction relation by the relation setting all cobordisms of genus bigger than one to zero. An illustration in the case of the unknot is given in Figure 5. □

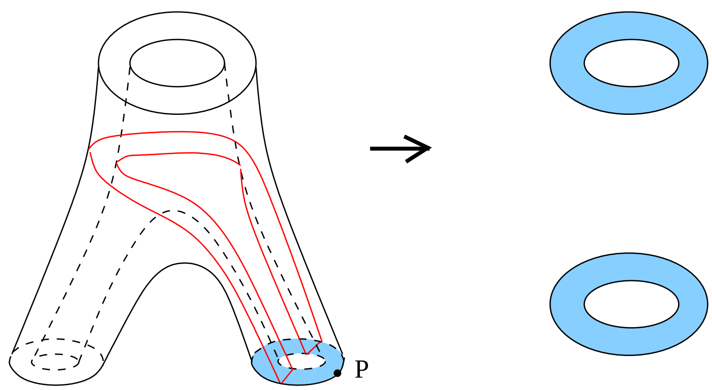


Figure 5 *Commutativity of diagrams (5).* The filled regions are annuli of genus 1. Applying the neck cutting relation to the red line, and removing components of genus 2, we get one half of the two genus 1 annuli, shown on the right.

Proof of Theorem 4. ¿From Theorems 5.1, 5.2 we know that saddles induce chain transformations. It remains to show that the morphisms associated to cups and caps do this also. The case of cups is easy, so we only discuss the case of caps. Let D and D_0 be two link diagrams which are related by a cap cobordism. Let K be the component of D_0 which is not contained in D. To show that $\iota^{\mathbf{s}_0,\mathbf{s}} : \mathrm{Kh}(D^{\mathbf{s}}) \to \mathrm{Kh}(D_0^{\mathbf{s}_0})$ induces a chain transformation, we write the differential of $[\![D_{0,\mathbf{n}_0}]\!]_{\alpha,\beta}$ as a sum $d_0 = d_0' + d_0''$, where d_0' is the sum of all morphisms $\mathrm{Kh}(e)$ which increase the number of pairs on K, and d_0'' is the sum of all morphisms $\mathrm{Kh}(e)$ which increase the number of pairs on one of the other components of D_0. It is easy to see that d_0'' commutes with ι. For $\alpha = 0$, $\beta = 1$, $\mathcal{F}_{\mathrm{Kh}}(d_0')\mathcal{F}_{\mathrm{Kh}}(\iota) = 0$, because the genus of the composition is bigger than one. To complete the proof, we show that for $\alpha = 1$ and $\beta = 1$, $\mathcal{F}_{\mathrm{Lee}}(d_0')\mathcal{F}_{\mathrm{Lee}}(\iota) = 0$. Note that $d_0' = \sum_{i=1}^{n} \mathrm{Kh}(e_i)$, where $\mathrm{Kh}(e_i)$ is the morphism $(1 + X(K,i))$ composed with an annulus glued to the strands i and $i+1$ of the n–cable of K. As in the proof of Theorem 5.1, we can replace $X(K,i)$ by $-X(K,i+1)$. Recall that $\iota^{\mathbf{s}_0,\mathbf{s}}$ is given by a union of cap cobordisms composed with the morphism $\varphi = \sum_{j=0}^{n} A_j \circ B_j$. Using the genus reduction relation, we obtain $\mathcal{F}_{\text{Lee}}((1+X(K,i)) \circ A_j) = 0$ for $i \leq j$ and $\mathcal{F}_{\text{Lee}}((1-X(K,i+1)) \circ B_j) = 0$ for $i \geq j$. Therefore $\mathcal{F}_{\text{Lee}}(\text{Kh}(e_i) \circ A_j \circ B_j) = 0$ for all i, j, and hence $\mathcal{F}_{\text{Lee}}(d_0')\mathcal{F}_{\text{Lee}}(\iota) = 0$. □

Remark 2. We do not know how to extend the original colored Khovanov bracket $[\![D_{\mathbf{n}}]\!]$ to a functor from the category $\mathcal{C}\mathbf{ob}^4_f$ to the category $\text{Kom}(\text{Kom}_{/h}(\mathcal{A}))$. For this bracket, the morphisms $\psi_{0,2}^{\mathbf{s}_0,\mathbf{s}}$ induce chain transformations (compare [7]), but there is no choice of γ, δ for which the morphisms $\bar{\psi}_{\gamma,\delta}^{\mathbf{s}_0,\mathbf{s}}$ induce chain transformations.

6. Rasmussen invariant for links

6.1. **Definition.** By Lee's theorem [8], the homology of $\mathcal{F}_{\text{Lee}}([\![L]\!])$ has rank $2^{|L|}$, where $|L|$ is the number of components of L. The generators of Lee homology are in bijection with the orientations of L. Hence, Lee homology of a knot has two generators. In particular, the graded module associated to the Lee homology of a knot has two homogeneous generators, whose topological degrees we denote by $s_{\max}$ and $s_{\min}$. The Rasmussen invariant $s(K)$ of a knot K is

$$s(K) := \frac{s_{\max} + s_{\min}}{2} .$$

Let us extend the Rasmussen construction to links. Let L be an *oriented* link. Let $\mathfrak{s}_o$ and $\mathfrak{s}_{\bar{o}}$ be the generators of the Lee homology corresponding to the orientation of L and the opposite orientation, respectively. Then by Lemma 3.5 in [10], the filtered topological degrees of $\mathfrak{s}_o + \mathfrak{s}_{\bar{o}}$ and $\mathfrak{s}_o - \mathfrak{s}_{\bar{o}}$ differ by two modulo 4. Further, we can show that they differ by exactly two. Indeed, a genus 1 cylinder cobordism induces an automorphism of $\mathcal{F}_{\text{Lee}}([\![L]\!])$ of topological degree -2, which interchanges $\mathfrak{s}_o + \mathfrak{s}_{\bar{o}}$ and $\mathfrak{s}_o - \mathfrak{s}_{\bar{o}}$. The Rasmussen invariant $s(L)$ of a link L is

$$s(L) := \frac{\deg(\mathfrak{s}_o + \mathfrak{s}_{\bar{o}}) + \deg(\mathfrak{s}_o - \mathfrak{s}_{\bar{o}})}{2}.$$

Note that $s(L) = \min(\deg(\mathfrak{s}_o + \mathfrak{s}_{\bar{o}}), \deg(\mathfrak{s}_o - \mathfrak{s}_{\bar{o}})) + 1$ and that the Rasmussen invariant of the n–component unlink is $1 - n$.

6.2. **Properties.** Let S be a smooth oriented cobordism from L_1 to L_2 such that every connected component of S has boundary in L_1. We will always assume that the orientations of L_1 and L_2 coincide with ones induced by S, in the sense that $\partial S = -L_1 \amalg L_2$. Then the Rasmussen estimate generalizes to

$$|s(L_2) - s(L_1)| \leq -\chi(S) \tag{7}$$

where $\chi(S)$ is the Euler characteristic of S. Indeed, arguing as in [10] we obtain the estimate $s(L_2) \geq s(L_1) + \chi(S)$. By reflecting $S \subset \mathbb{R}^3 \times [0,1]$ along $\mathbb{R}^3 \times \{1/2\}$, we obtain a cobordism from L_2 to L_1 with the same Euler characteristic as S. This gives us the estimate $s(L_1) \geq s(L_2) + \chi(S)$.

Lemma 6.1. *Let $\bar{L}$ be the mirror image of L and $\#$, $\amalg$ denote the connected sum and the disjoint union, respectively. Then*

$$s(L_1 \amalg L_2) = s(L_1) + s(L_2) - 1 \tag{8}$$

$$s(L_1) + s(L_2) - 2 \leq s(L_1 \# L_2) \leq s(L_1) + s(L_2) \tag{9}$$

$$-2|L| + 2 \leq s(L) + s(\bar{L}) \leq 2 \tag{10}$$

Note that the first inequality of (10) becomes an equality if L is an unlink.

Proof of Lemma 6.1. Let o_1, o_2 and o denote the orientations of L_1, L_2 and $L_1 \amalg L_2$, respectively.

The filtered modules $\mathcal{F}_{\mathrm{Lee}}([\![L_1 \amalg L_2]\!])$ and $\mathcal{F}_{\mathrm{Lee}}([\![L_1]\!]) \otimes \mathcal{F}_{\mathrm{Lee}}([\![L_2]\!])$ are isomorphic by an isomorphism which sends $\mathfrak{s}_o$ to $\mathfrak{s}_{o_1} \otimes \mathfrak{s}_{o_2}$. Hence (8) follows from $\deg(\mathfrak{s}_o) = \min(\deg(\mathfrak{s}_o + \mathfrak{s}_{\bar{o}}), \deg(\mathfrak{s}_o - \mathfrak{s}_{\bar{o}})) = s(L_1 \amalg L_2) - 1$ and $\deg(\mathfrak{s}_{o_i}) = \min(\deg(\mathfrak{s}_{o_i} + \mathfrak{s}_{\bar{o}_i}), \deg(\mathfrak{s}_{o_i} - \mathfrak{s}_{\bar{o}_i})) = s(L_i) - 1$ (cf. [10, Corollary 3.6]). (9) follows from (7) and (8) because $L_1 \amalg L_2$ and $L_1 \# L_2$ are related by a saddle cobordism. Similarly, (10) can be deduced from (7) and (8) because there is a cobordism, consisting of $|L|$ saddle cobordisms, which connects $L \amalg \bar{L}$ to the $|L|$–component unlink. □

6.3. **Obstructions to sliceness.** The notion of sliceness admits different generalizations to links. We say that an oriented link L is slice in *the weak sense* if there exists an oriented smooth connected surface $P \subset B^4$ of genus zero, such that $\partial P = L$. L is slice in *the strong sense* if every component bounds a disk in B^4 and all these disks are disjoint. Recently, Cimasoni and Florens [5] unified different notions of sliceness by introducing colored links.

The Rasmussen invariant of links is an obstruction to sliceness.

Lemma 6.2. *Let L be slice in the weak sense, then*

$$|s(L)| \leq |L| - 1.$$

Proof. If L is slice in the weak sense, then there exist an oriented genus 0 cobordism from L to the unknot. Applying (7) to this cobordism we get the result. □

The multivariable Levine–Tristram signature defined in [5] is also an obstruction to sliceness. However, for knots with the trivial Alexander polynomial, the Levine–Tristram signature is constant and equal to the ordinary signature. Therefore, for a disjoint union of such knots the Rasmussen link invariant is often a better obstruction than the multivariable signature. Using the Shumakovitch list of knots with the trivial Alexander polynomial, but nontrivial Rasmussen invariant [11] and *Knotscape*, one can easily construct examples. E.g. the multivariable signature of $K_{15n_{28998}} \amalg K_{15n_{40132}} \amalg K_{13n_{1496}}$ vanishes identically, however $s(K_{15n_{28998}} \amalg K_{15n_{40132}} \amalg K_{13n_{1496}}) = 4 > 3 - 1$, hence this split link is not slice in the weak sense. Similarly, the Rasmussen invariant, but not the signature, is an obstruction to sliceness for the following split links: $K_{15n_{113775}} \amalg K_{14n_{7708}}$, $K_{15n_{58433}} \amalg K_{15n_{58501}}$, etc.

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