A REMARK ON THE TOPOLOGY OF (n,n) SPRINGER VARIETIES

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Abstract. We prove a conjecture of Khovanov [Kho04] which identifies the topological space underlying the Springer variety of complete flags in $\mathbb{C}^{2n}$ stabilized by a fixed nilpotent operator with two Jordan blocks of size n.

1. Introduction

Let E_n be a complex vector space of dimension $2n$ and $z_n\colon E_n \to E_n$ a nilpotent linear endomorphism with two nilpotent Jordan blocks, each of them of size n. A *complete flag* in E_n is an ascending sequence of linear subspaces $0 \subsetneq L_1 \subsetneq L_2 \subsetneq \ldots \subsetneq L_{2n} = E_n$. The (n,n) *Springer variety* is the set

$$\mathfrak{B}_{n,n} := \{\text{complete flags in } E_n \text{ stabilized by } z_n\},$$

where a complete flag is said to be *stabilized* by z_n if each of the subspaces L_j is stable under z_n, i.e. if $z_n L_j \subset L_j$ for all $j \in \{1, \ldots, 2n\}$.

It is known that $\mathfrak{B}_{n,n}$ is a complex projective variety of (complex) dimension n, and that the irreducible components of $\mathfrak{B}_{n,n}$ are topologically trivial (but algebraically non-trivial) iterated $\mathbb{P}^1$-bundles over a point (where $\mathbb{P}^1$ is the complex projective line, i.e., topologically, $\mathbb{P}^1 \cong S^2$). Moreover, a result of Fung [Fun02] (going back to earlier work of Spaltenstein [Spa76] and Vargas [Var79]), describes the irreducible components of $\mathfrak{B}_{n,n}$ explicitly in terms of crossingless matchings of $2n$ points:

Proposition 1.1 (Fung)**.** *The irreducible components of $\mathfrak{B}_{n,n}$ are parametrized by crossingless matchings of $2n$ points. Furthermore, the irreducible component K_a associated to $a \in B^n$ can be described explicitly, as follows:*

$$K_a = \{(L_1, \ldots, L_{2n}) \in \mathfrak{B}_{n,n}\colon L_{s_a(j)} = z_n^{-d_a(j)} L_{j-1}\, \forall j \in O_a\}$$

Here, B^n is the set of all crossingless matchings of $2n$ points. Elements of B^n can be thought of as diagrams consisting of n disjoint, nested cups, as in Figure 1. Equivalently, elements of B^n are partitions of the set $\{1, 2, \ldots, 2n\}$ into pairs, such that there is no quadruple $i < j < k < l$ with (i,k) and (j,l) paired. For an element $a \in B^n$, we denote by O_a the set of all i appearing in a pair $(i,j) \in a$ with $i < j$; and if $(i,j) \in a$ is a pair with $i < j$, then we define $s_a(i) := j$ and $d_a(i) := (s_a(i) - i + 1)/2$. Note that $d_a(i)$ is always an integer because $s_a(i) - i - 1$ is twice the number of cups that are contained strictly inside the cup with endpoints i and $s_a(i)$.

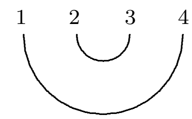

FIGURE 1. Crossingless matching $\{(1,4),(2,3)\}$.

In [Kho04], Khovanov proved that the integer cohomology ring of $\mathfrak{B}_{n,n}$ is isomorphic to the center of the ring $H^n = \bigoplus_{a,b\in B^n} {}_b(H^n)_a$, defined in [Kho02]. To show this, Khovanov first proved that $\mathfrak{B}_{n,n}$ has the same integer cohomology ring as a topological space $\widetilde{S} \subset (\mathbb{P}^1)^{2n} = \mathbb{P}^1 \times \ldots \times \mathbb{P}^1$ ($2n$ factors), defined by $\widetilde{S} := \bigcup_{a\in B^n} S_a \subset (\mathbb{P}^1)^{2n}$, where

$$S_a := \{(l_1,\ldots,l_{2n}) \in (\mathbb{P}^1)^{2n} \colon l_j = l_{s_a(j)}\, \forall j \in O_a\}.$$

The goal of this paper is to show the following stronger statement, which was also conjectured by Khovanov ([Kho04, Conjecture 1]):

Theorem 1.2. *$\mathfrak{B}_{n,n}$ and $\widetilde{S}$ are homeomorphic.*

Our proof of Theorem 1.2 is based on Proposition 1.1 and on the observation of Cautis and Kamnitzer [CK07] that $\mathfrak{B}_{n,n}$ can be embedded into a (smooth) complex projective variety Y_{2n} diffeomorphic to $(\mathbb{P}^1)^{2n}$. Besides the diffeomorphism

$$\phi_{2n} \colon Y_{2n} \longrightarrow (\mathbb{P}^1)^{2n}$$

of Cautis and Kamnitzer, whose definition we review in Section 2, we will need an involutive diffeomorphism

$$I_{2n} \colon (\mathbb{P}^1)^{2n} \longrightarrow (\mathbb{P}^1)^{2n}$$

defined by $I_{2n}(l_1,\ldots,l_{2n}) := (l'_1,\ldots,l'_{2n})$ with

$$l'_j := \begin{cases} l_j & \text{if } j \text{ is odd,} \\ l_j^{\perp} & \text{if } j \text{ is even,} \end{cases}$$

where $l_j^{\perp} \subset \mathbb{C}^2$ is the orthogonal complement (w.r.t. the standard hermitian product on $\mathbb{C}^2$) of the complex line $l_j \subset \mathbb{C}^2$ (or, equivalently, the antipode of the point $l_j \in \mathbb{P}^1 \cong S^2$). In Section 3, we prove the following result, which implies Theorem 1.2:

Proposition 1.3. *The diffeomorphism $I_{2n} \circ \phi_{2n}$ maps $K_a \subset Y_{2n}$ to $S_a \subset (\mathbb{P}^1)^{2n}$ for all $a \in B^n$, and hence $\mathfrak{B}_{n,n}$ to $\widetilde{S}$.*

The author had the main idea for this article in Spring 2007 while he was preparing a talk for an informal seminar on link homology and coherent sheaves organized by Mikhail Khovanov at Columbia University. In a recent article [RT08], Russell and Tymoczko studied an action of the symmetric group S_{2n} on the cohomology ring of $\mathfrak{B}_{n,n}$. In this context, they also proved Theorem 1.2. Although our proof is similar to theirs, our work is completely independent.

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2. Diffeomorphism ϕ_m

In the following, E is the complex vector space $E := \mathbb{C}^N \oplus \mathbb{C}^N$ (for some $N > 0$), and $z\colon E \to E$ is the nilpotent linear endomorphism given by $ze_j := e_{j-1}$ and $zf_j := f_{j-1}$ for all $j \in \{2, \ldots, N\}$, and $ze_1 := zf_1 := 0$, where $\{e_1, \ldots, e_N\}$ is the standard basis for the first $\mathbb{C}^N$ summand in E, and $\{f_1, \ldots, f_N\}$ is the standard basis of the second $\mathbb{C}^N$ summand in E. For $n \leq N$, we denote by $E_n \subset E$ the subspace $E_n := \mathbb{C}^n \oplus \mathbb{C}^n = \operatorname{span}(e_1, \ldots, e_n) \oplus \operatorname{span}(f_1, \ldots, f_n)$, or equivalently, $E_n = z^{-n}(0) = \ker(z^n) = \operatorname{im}(z^{N-n})$, and we denote by $\langle ., . \rangle_E$ the standard hermitian product on E, satisfying

$$\langle e_i, e_j \rangle_E := \langle f_i, f_j \rangle_E := \delta_{i,j} \quad , \quad \langle e_i, f_j \rangle_E := 0,$$

for all $i, j \in \{1, \ldots, N\}$, and by $\langle ., . \rangle$ the standard hermitian product on $\mathbb{C}^2$, satisfying

$$\langle e, e \rangle := \langle f, f \rangle := 1 \quad , \quad \langle e, f \rangle := 0,$$

where $\{e, f\}$ is the standard basis of $\mathbb{C}^2$.

2.1. **Stable subspaces.** A subspace $W \subset E$ is called *stable* under z if it satisfies $zW \subset W$. Note that this condition also implies $z^2 W \subset zW$ and $W \subset z^{-1}W$, so if W is stable under z, then so are its images and preimages under z. Moreover, if a stable subspace W satisfies $W \subset \operatorname{im}(z)$, then $z\colon z^{-1}W \to W$ is surjective and therefore

$$\dim((z^{-1}W) \cap W^\perp) = \dim(z^{-1}W/W) = \dim(z^{-1}W) - \dim(W) = \dim(E_1) = 2$$

where we have used that $z^{-1}W \supset z^{-1}(0) = \ker(z) = E_1$. Let $C\colon E \to \mathbb{C}^2$ be the linear map defined by $C(e_j) := e$ and $C(f_j) := f$ for all $j \in \{1, \ldots, N\}$. The following lemma is taken from [CK07, Lemma 2.2]:

Lemma 2.1. *If $W \subset E$ is stable under z and contained in $\operatorname{im}(z)$, then the restriction $C|_{(z^{-1}W)\cap W^\perp}\colon (z^{-1}W) \cap W^\perp \to \mathbb{C}^2$ is an isomotric isomorphism.*

For the convenience of the reader, we recall the proof given in [CK07].

Proof. Since $(z^{-1}W) \cap W^\perp$ is two-dimensional, it suffices to show that the restriction of C to $(z^{-1}W) \cap W^\perp$ is an isometry. For this, let $v, w \in (z^{-1}W) \cap W^\perp$ with $v = v_1 + \ldots + v_N$ and $w = w_1 + \ldots + w_N$ where $v_j, w_j \in \operatorname{span}(e_j, f_j)$. Then we have

$$\langle v, w \rangle_E = \sum_i \langle v_i, w_i \rangle_E = \sum_i \langle C(v_i), C(w_i) \rangle$$

and

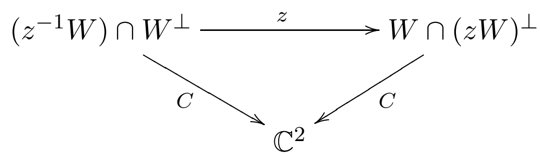
$$\langle C(v), C(w) \rangle = \langle \sum_i C(v_i), \sum_j C(w_j) \rangle = \sum_{i,j} \langle C(v_i), C(w_j) \rangle.$$

To prove that the restriction of C to $(zW) \cap W^\perp$ is an isometry, i.e. that $\langle v, w \rangle_E = \langle C(v), C(w) \rangle$, we must therefore show $\sum_{i \neq j} \langle C(v_i), C(w_j) \rangle = 0$. We will actually prove a stronger statement, namely that $\sum_{i=j+k} \langle C(v_i), C(w_j) \rangle = 0$ for each fixed $k \neq 0$. Assuming $k > 0$ (the case $k < 0$ being similar), we can write

$$\sum_{i=j+k} \langle C(v_i), C(w_j) \rangle = \sum_{i=j+k} \langle v_i, w_j \rangle_E = \langle v, z^k w \rangle_E,$$

and since $v, w \in (z^{-1}W) \cap W^{\perp}$, we have $v \in W^{\perp}$ and $z^k w \in z^k(z^{-1}W) \subset z^{k-1}W \subset W$, whence $\langle v, z^k w\rangle_E = 0$, as desired. □

Lemma 2.2. *Let $W \subset E$ be a stable subspace such that $\ker(z) \subset W \subset \operatorname{im}(z)$. Then z maps $W^{\perp} \cap z^{-1}W$ isomorphically to $(zW)^{\perp} \cap W$, and the following diagram commutes:*



Proof. It is apparent that $W \cap (zW)^{\perp} \cong W/(zW)$ is two-dimensional, and, by the previous lemma, C restricts to an isomorphism on $(z^{-1}W) \cap W^{\perp}$, so we only need to prove that z maps elements of $(z^{-1}W) \cap W^{\perp}$ to elements of $W \cap (zW)^{\perp}$, and that the above diagram commutes. Thus, let $v \in (z^{-1}W) \cap W^{\perp}$, and write v as

$$v = v_1 + \ldots + v_N$$

for $v_j \in \operatorname{span}(e_j, f_j)$. Since $v \in W^{\perp}$ and $W \supset \ker(z) = E_1 = \operatorname{span}(e_1, f_1)$, we have $v_1 = 0$, and since $C(zv_j) = C(v_j)$ for all $j \geq 2$, this implies $C(zv) = C(v)$. We clearly have $zv \in W$ (because $v \in z^{-1}W$), so the only thing that remains to be shown is that $zv \in (zW)^{\perp}$. For this, consider any $w \in W$ and write w as $w = w_1 + \ldots + w_N$ for $w_j \in \operatorname{span}(e_j, f_j)$. Since $\langle zv_j, zw_j\rangle_E = \langle v_j, w_j\rangle_E$ for all $j \geq 2$, and since $v_1 = 0$ and $v \in W^{\perp}$, we see that $\langle zv, zw\rangle_E = \langle v, w\rangle_E = 0$, and thus $zv \in (zW)^{\perp}$. □

2.2. **Y_m and ϕ_m.** For $m \leq N$, Cautis and Kamnitzer [CK07, Section 2] define a complex projective variety Y_m,

$$Y_m := \{(L_1, \ldots, L_m) \in F_m : \dim(L_j) = j \text{ and } zL_j \subset L_j \,\forall j\},$$

where F_m is the set of all partial flags $0 \subsetneq L_1 \subsetneq L_2 \subsetneq \ldots \subsetneq L_m \subset E$. Note that the conditions $zL_j \subset L_j$ and $zL_{j-1} \subset L_{j-1}$ imply that z descends to an endomorphism of L_j/L_{j-1}, and since L_j/L_{j-1} is one-dimensional and z nilpotent, this endomorphism must be the zero-map, so the spaces L_j in $(L_1, \ldots, L_m) \in Y_m$ actually satisfy the seemingly stronger condition $zL_j \subset L_{j-1}$. In particular, $L_m \subset z^{-1}L_{m-1} \subset z^{-2}L_{m-2} \subset \ldots \subset z^{-m}(0) = \ker(z^m) = E_m$, so as far as the definition of Y_m is concerned, we could restrict ourselves to the space $E_m = \mathbb{C}^m \oplus \mathbb{C}^m$ instead of working with the bigger space $E = \mathbb{C}^N \oplus \mathbb{C}^N$. In particular, Y_m is independent of the choice of N (as long as $N \geq m$).

Note also that the assignment $(L_1, \ldots, L_{m-1}, L_m) \mapsto (L_1, \ldots, L_{m-1})$ defines a $\mathbb{P}^1$-bundle $Y_m \to Y_{m-1}$. Indeed, a point in the fiber above $(L_1, \ldots, L_{m-1}) \in Y_{m-1}$ is obtained from $(L_1, \ldots, L_{m-1})$ by choosing an L_m such that $L_{m-1} \subset L_m \subset z^{-1}L_{m-1}$, and since $z^{-1}L_{m-1}/L_{m-1}$ is two-dimensional, we have a $\mathbb{P}^1$ worth of choices. Denoting by $L_{j-1}^{\perp}$ the orthogonal complement of L_{j-1} w.r.t. $\langle .,.\rangle_E$, we can identify $z^{-1}L_{m-1}/L_{m-1}$ with $(z^{-1}L_{m-1}) \cap L_{m-1}^{\perp}$, and by Lemma 2.1, the map $C\colon E \to \mathbb{C}^2$ identifies $(z^{-1}L_{m-1}) \cap L_{m-1}^{\perp}$ with $\mathbb{C}^2$. Therefore, the $\mathbb{P}^1$-bundle $Y_m \to Y_{m-1}$ is topologically trivial (i.e., topologically, $Y_m \cong \mathbb{P}^1 \times Y_{m-1}$), and Cautis and Kamnitzer use this to define a diffeomorphism

$$\phi_m \colon Y_m \longrightarrow (\mathbb{P}^1)^m$$

by $\phi_m(L_1, \ldots, L_m) := (C(L_1), C(L_2 \cap L_1^\perp), C(L_3 \cap L_2^\perp), \ldots, C(L_m \cap L_{m-1}^\perp))$.

2.3. **Subvarieties $X_{m,i} \subset Y_m$.** For each $i \in \{1, \ldots, m-1\}$, Cautis and Kamnitzer [CK07, Section 2] define a subvariety $X_{m,i} \subset Y_m$,

$$X_{m,i} := \{(L_1, \ldots, L_m) \in Y_m : \ L_{i+1} = z^{-1}(L_{i-1})\},$$

together with a surjection

$$q_{m,i} \colon X_{m,i} \longrightarrow Y_{m-2},$$

given by $q_{m,i}(L_1, \ldots, L_m) := (L_1, \ldots, L_{i-1}, zL_{i+2}, \ldots, zL_m) \in Y_{m-2}$. The following (easy) Lemma was shown in [CK07, Theorem 2.1].

Lemma 2.3. *The map $\phi_m \colon Y_m \to (\mathbb{P}^1)^m$ takes $X_{i,m}$ diffeomorphically to*

$$A_{m,i} := \{(l_1, \ldots, l_m) \in (\mathbb{P}^1)^m : \ l_{i+1} = l_i^\perp\},$$

where $l_i^\perp$ denotes the orthogonal complement of the line $l_i \subset \mathbb{C}^2$ w.r.t. $\langle ., . \rangle$.

Let $f_{m,i} \colon (\mathbb{P}^1)^m \to (\mathbb{P}^1)^{m-2}$ be the forgetful map sending $(l_1, \ldots, l_m) \in (\mathbb{P}^1)^m$ to $(l_1, \ldots, l_{i-1}, l_{i+2}, \ldots, l_m) \in (\mathbb{P}^1)^{m-2}$, and let

$$g_{m,i} : A_{m,i} \longrightarrow (\mathbb{P}^1)^{m-2}$$

be the restriction of $f_{m,i}$ to $A_{m,i}$.

Lemma 2.4. *Let $\psi_{m,i} \colon X_{m,i} \to A_{m,i}$ be the restriction of ϕ_m to $X_{m,i} \subset Y_m$. Then the following diagram commutes:*

$$\begin{array}{ccc} X_{m,i} & \xrightarrow{q_{m,i}} & Y_{m-2} \\ \downarrow{\scriptstyle \psi_{m,i}} & & \downarrow{\scriptstyle \phi_{m-2}} \\ A_{m,i} & \xrightarrow{g_{m,i}} & (\mathbb{P}^1)^{m-2} \end{array}$$

Proof. It is straightforward to check that $g_{m,i} \circ \psi_m$ maps $(L_1, \ldots, L_m) \in X_{m,i}$ to the tuple $(l'_1, \ldots, l'_{m-2}) \in (\mathbb{P}^1)^{m-2}$, where

$$l'_j = \begin{cases} C(L_j \cap L_{j-1}^\perp) & \text{if } j < i, \\ C(L_{j+2} \cap L_{j+1}^\perp) & \text{if } j \geq i, \end{cases}$$

and $\phi_{m-2} \circ q_{m,i}$ maps $(L_1, \ldots, L_m) \in X_{m,i}$ to the tuple $(l''_1, \ldots, l''_{m-2}) \in (\mathbb{P}^1)^{m-2}$, where

$$l''_j = \begin{cases} C(L_j \cap L_{j-1}^\perp) & \text{if } j < i, \\ C(zL_{j+2} \cap (zL_{j+1})^\perp) & \text{if } j \geq i. \end{cases}$$

To prove $g_{m,i} \circ \psi_m = \phi_{m-2} \circ q_{m,i}$, we must therefore show that

$$C(L_{j+2} \cap L_{j+1}^\perp) = C(zL_{j+2} \cap (zL_{j+1})^\perp)$$

holds for all $j \geq i$. But if $j \geq i$, then $L_{j+1} \supset L_{i+1} = z^{-1}L_{i-1} \supset z^{-1}(0) = \ker(z)$, and (by increasing N if necessary) we can also assume that $L_{j+1} \subset \operatorname{im}(z)$. Thus, Lemma 2.2 applied to $W := L_{j+1}$ tells us that z maps $(z^{-1}W)\cap W^\perp$ to $W\cap(zW)^\perp$, and that $C(v) = C(zv)$ for all $v \in (z^{-1}W)\cap W^\perp$. Now the equality $C(L_{j+2}\cap L_{j+1}^\perp) = C(zL_{j+2} \cap (zL_{j+1})^\perp)$ follows because z maps $L_{j+2} \cap L_{j+1}^\perp \subset (z^{-1}W) \cap W^\perp$ to $zL_{j+2} \cap (zL_{j+1})^\perp \subset W \cap (zW)^\perp$. □

3. Proof of Proposition 1.3

In this section, we use the same notations as before, except that we now assume $m = 2n$ (and hence $N \geq 2n$). Then the Springer variety $\mathfrak{B}_{n,n}$ is naturally contained in Y_{2n} as

$$\mathfrak{B}_{n,n} := \{(L_1, \ldots, L_{2n}) \in Y_{2n} :\ L_{2n} = E_n\},$$

where $E_n := \operatorname{span}(e_1, \ldots, e_n) \oplus \operatorname{span}(f_1, \ldots, f_n)$, and Proposition 1.1 tells us that the irreducible component $K_a \subset \mathfrak{B}_{n,n} \subset Y_{2n}$ associated to the crossingless matching $a \in B^n$ is equal to the set of all $(L_1, \ldots, L_{2n}) \in Y_{2n}$ satisfying

$$L_{s_a(j)} = z_n^{-d_a(j)} L_{j-1}$$

for all $j \in O_a$, where $z_n \colon E_n \to E_n$ is the restriction of z to E_n. A priori, $z_n^{-d_a(j)} L_{j-1}$ could a priori be a proper subspace of $z^{-d_a(j)}L_{j-1}$ (because $z^{-d_a(j)}L_{j-1}$ might not be contained in E_n), but it turns out that $z_n^{-d_a(j)}L_{j-1}$ is equal to $z^{-d_a(j)}L_{j-1}$ whenever $(L_1, \ldots, L_{2n}) \in K_a$. In fact, we have:

Lemma 3.1. $K_a = \{(L_1, \ldots, L_{2n}) \in Y_{2n} :\ L_{s_a(j)} = z^{-d_a(j)}L_{j-1}\, \forall j \in O_a\}$.

Proof. Suppose $(L_1, \ldots, L_{2n})$ is contained in K_a. Then the condition $z_n^{-d_a(j)}L_{j-1} = L_{s_a(j)}$, combined with $\dim(L_{j-1}) = j - 1$, $\dim(L_{s_a(j)}) = s_a(j)$, and $\dim(\ker(z)) = 2$, implies

$$\begin{aligned}\dim(z^{-d_a(j)}L_{j-1}) &= 2d_a(j) + \dim(L_{j-1}) = 2d_a(j) + j - 1 = s_a(j)\\ &= \dim(L_{s_a(j)}) = \dim(z_n^{-d_a(j)}L_{j-1}),\end{aligned}$$

and thus $z^{-d_a(j)}L_{j-1} = z_n^{-d_a(j)}L_{j-1}$. Conversely, suppose $(L_1, \ldots, L_{2n}) \in Y_{2n}$ satisfies $z^{-d_a(j)}L_{j-1} = L_{s_a(j)}$ for all $j \in O_a$. Then we must show that $L_{2n} = E_n$. To prove this, let us call a pair $(k, l) \in a$ *outermost* if there is no pair $(k', l') \in a$ such that $k' < k < l < l'$. Then the outermost pairs in a form a sequence $(k_1, l_1), (k_2, l_2), \ldots, (k_r, l_r) \in a$ such that $k_1 = 1$, $l_r = 2n$, and $k_{s+1} = l_s + 1$ for all $s < r$, and $d_a(k_1) + \ldots + d_a(k_r) = n$. Using $z^{-d_a(j)}L_{j-1} = L_{s_a(j)}$ successively for $j \in \{k_r, k_{r-1}, \ldots, k_1\} \subset O_a$, we obtain

$$L_{2n} = z^{-d_a(k_r)}L_{l_{r-1}} = z^{-d_a(k_r)}z^{-d_a(k_{r-1})}L_{l_{r-2}} = \ldots = z^{-n}(0) = E_n,$$

as desired. □

From now on, $a \in B^n$ is a fixed crossingless matching of $2n$ points, and i is an index such that $s_a(i) = i + 1$, i.e., such that $(i, i + 1)$ is a pair in a. We denote by $a' \in B^{n-1}$ the crossingless matching obtained from a by removing the pair $(i, i + 1)$ (and renumbering indices $j \geq i + 2$ such that $j \in \{i + 2, \ldots, 2n\}$ becomes $j - 2 \in \{i, \ldots, 2n - 2\}$), and by q the map $q_{2n,i} \colon X_{2n,i} \to Y_{2n-2}$, defined as in the previous section.

Lemma 3.2. $K_a = q^{-1}(K_{a'})$.

Proof. Since $s_a(i) = i+1$ and $d_a(i) = (s_a(i) - i + 1)/2 = 1$, the equality $L_{i+1} = z^{-1}L_{i-1}$ holds for each $(L_1, \dots, L_{2n}) \in K_a$, and thus $K_a \subset Y_{2n}$ is contained in $X_{2n,i}$. It remains to show that an element $(L_1, \dots, L_{2n}) \in X_{2n,i}$ satisfies the conditions $L_{s_a j} = z^{-d_a(j)}L_{j-1}$ for all $j \in O_a \setminus \{i\}$ if and only if the element $(L'_1, \dots, L'_{2n-2}) := q(L_1, \dots, L_{2n}) = (L_1, \dots, L_{i-1}, zL_{i+2}, \dots, zL_{2n}) \in Y_{2n-2}$ satisfies the conditions $L'_{s_{a'}(j)} = z^{-d_{a'}(j)}L'_{j-1}$ for all $j \in O_{a'}$. We divide the proof into three cases.

Case 1. If $j < s_a(j) < i$, then the equivalence

$$L_{s_a(j)} = z^{-d_a(j)}L_{j-1} \iff L'_{s_{a'}(j)} = z^{-d_{a'}(j)}L'_{j-1}$$

is obvious because the quantities appearing on either side of $\iff$ are identical.

Case 2. If $j < i < i+1 < s_a(j)$, then $L'_{j-1} = L_{j-1}$, $L'_{s_{a'}(j)} = zL_{s_a(j)}$, and $d_{a'}(j) = d_a(j) - 1$, so we must show:

$$L_{s_a(j)} = z^{-d_a(j)}L_{j-1} \iff zL_{s_a(j)} = z^{-d_a(j)+1}L_{j-1}$$

But this follows simply by applying z (resp., z^{-1}) to the equalities on either side of $\iff$, and observing that $z^{-1}(zL_{s_a(j)}) = L_{s_a(j)}$ (because $L_{s_a(j)} \supset L_{i+1} = z^{-1}L_{i-1} \supset z^{-1}(0) = \ker(z)$), and that $z(z^{-d_a(j)}L_{j-1}) = z^{-d_a(j)+1}L_{j-1}$ (because, by increasing N if necessary, we may assume $z^{-d_a(j)+1}L_{j-1} \subset \mathrm{im}(z)$).

Case 3. If $i+1 < j < s_a(j)$, then $L'_{j-3} = zL_{j-1}$, $L_{s_{a'}(j-2)} = zL_{s_a(j)}$, and $d_{a'}(j-2) = d_a(j)$, so we must show:

$$L_{s_a(j)} = z^{-d_a(j)}L_{j-1} \iff zL_{s_a(j)} = z^{-d_a(j)}zL_{j-1}$$

As in Case 2, this follows by applying z (resp., z^{-1}) to the equalities on either side of $\iff$. □

Note that (since $s_a(j) - j$ is odd for all $j \in O_a$) the involutive diffeomorphism $I_{2n} \colon (\mathbb{P}^1)^{2n} \to (\mathbb{P}^1)^{2n}$ defined in the introduction exchanges the subset $S_a \subset (\mathbb{P}^1)^{2n}$ with the subset

$$T_a := \{(l_1, \dots, l_{2n}) \in (\mathbb{P}^1)^{2n} : l_{s_a(j)} = l_j^{\perp}\ \forall j \in O_a\} \subset (\mathbb{P}^1)^{2n}$$

To prove Proposition 1.3, we must therefore show that ϕ_{2n} maps K_a to T_a for all $a \in B^n$. We will need the following lemma, in which a, i and a' are as in the previous lemma, and g denotes the map $g_{2n,i} \colon A_{2n,i} \to (\mathbb{P}^1)^{2n-2}$, defined as in the previous section.

Lemma 3.3. $T_a = g^{-1}(T_{a'})$.

Proof. This follows directly from the definitions of g, $A_{2n,i}$, T_a and $T_{a'}$. □

We are now ready to prove Proposition 1.3.

Proof of Proposition 1.3. Induction on n. The case $n = 1$ is trivial because the only crossingless matching of 2 points is $a_1 := \{(1,2)\}$, and $\phi_2 \colon Y_2 \to \mathbb{P}^1 \times \mathbb{P}^1$ maps $\mathfrak{B}_{1,1} = K_{a_1} = X_{2,1} \subset Y_2$ diffeomorphically to $T_{a_1} = A_{2,1} \subset \mathbb{P}^1 \times \mathbb{P}^1$.

Thus, let $n > 1$, and suppose we have already proven the proposition for $n-1$. Let $a \in B^n$. Then there is an $i \in \{1, \dots, 2n-1\}$ such that $s_a(i) = i+1$, i.e., such that $(i, i+1) \in a$. As above, we denote by $a' \in B^{n-1}$ the crossingless matching obtained from a by removing the pair $(i, i+1)$ (and renumbering all $j \geq i+2$), and by q (resp., g) the map $q_{2n,i}$ (resp., $g_{2n,i}$). By induction, we know that ϕ_{2n-2} maps $K_{a'}$ to $T_{a'}$, so Lemma 2.4 gives us the following commutative diagram:

$$\begin{array}{ccccccc}
q^{-1}(K_{a'}) & \hookrightarrow & X_{2n,i} & \xrightarrow{q} & Y_{2n-2} & \hookleftarrow & K_{a'} \\
\downarrow{\scriptstyle \psi_{2n,i}} & & \downarrow{\scriptstyle \psi_{2n,i}} & & \downarrow{\scriptstyle \phi_{2n-2}} & & \downarrow{\scriptstyle \phi_{2n-2}} \\
g^{-1}(T_{a'}) & \hookrightarrow & A_{2n,i} & \xrightarrow{g} & (\mathbb{P}^1)^{2n-2} & \hookleftarrow & T_{a'}
\end{array}$$

Hence we get $\psi_{2n,i}(q^{-1}(K_{a'})) = g^{-1}(T_{a'})$, and by Lemmas 3.2 and 3.3, this implies

$$\psi_{2n,i}(K_a) = T_a,$$

thus completing the inductive step. ☐

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