AN ELEMENTARY FACT ABOUT UNLINKED BRAID CLOSURES

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Abstract. Let $n \in \mathbb{Z}^+$. We provide a short Khovanov homology proof of the following classical fact: if the closure of an n–strand braid σ is the n–component unlink, then σ is the trivial braid.

Let $\mathcal{B}_n$ denote the n–strand braid group, $\mathbb{1}_n \in \mathcal{B}_n$ the n–strand trivial braid, and U_n the n–component unlink in S^3. Denote by $\widehat{\sigma}$ the closure of $\sigma \in \mathcal{B}_n$, considered as a link in S^3. The following fact first appears in the literature in [2, Thm. 4.1]:

Proposition 1. *Let $\sigma \in B_n$. If $\widehat{\sigma} = U_n$, then $\sigma = \mathbb{1}_n$.*

The purpose of this note is to provide a short Khovanov homology proof of Proposition 1. Although the classical proof contained in [2] is straightforward, we hope the Khovanov homology proof will also be of interest, since it suggests ways in which algebraic properties of Khovanov homology–in particular, its module structure–can give information about braid dynamics.

It may be of interest to the reader that there is a parallel story–going through the double-branched cover operation–involving minimal complexity fibered links in connected sums of copies of $S^1 \times S^2$.

Explicitly, let Y_n denote $\#^n(S^1 \times S^2)$. For L a fibered link with fiber F, we will abuse terminology and refer to $\chi(F)$ as the *Euler characteristic of* L.

Define

$$\mathcal{L}_n := \{\ell \in \mathbb{Z}^+ \mid \ell \leq (n+1) \text{ and } \ell \equiv (n+1) \mod 2\}.$$

Note that for each $\ell \in \mathcal{L}_n$, it is straightforward to construct a fibered link, $\mathbf{L}_\ell \subset Y_n$, of Euler characteristic $1-n$. See Figure 1. The monodromy of $\mathbf{L}_\ell$ is trivial, and the pair $(Y_n, \mathbf{L}_\ell)$ is well-defined up to diffeomorphism.

The following result appears in [11]. Indeed, after the first version of this note appeared, it was pointed out in [3, Cor. 1.3] that Proposition 2 implies Proposition 1.

Proposition 2. [11, Prf. of Thm. 1.3] *Let $L_\ell \subset Y_n$ be a fibered, ℓ–component link with $\ell \in \mathcal{L}_n$ and Euler characteristic $1-n$. Then the pair (Y_n, L_ℓ) is diffeomorphic to the pair $(Y_n, \mathbf{L}_\ell)$.*

It is clear (cf. Lemma 1) that if $\ell \notin \mathcal{L}_n$, then an ℓ–component link cannot have Euler characteristic $1-n$. It is also clear (cf. Lemma 2) that $1-n$ is the maximal possible Euler characteristic among all fibered links in Y_n. Informally, Proposition 2 therefore says that for allowable ℓ, maximal Euler characteristic fibered ℓ–component links in $\#^n(S^1 \times S^2)$ are unique up to diffeomorphism.

In Section 2.1 we will give an alternative proof of Proposition 2 that is formally analogous to the Khovanov homology proof of Proposition 1.

We thank John Baldwin for pointing out that (this proof of) Proposition 2 implies:

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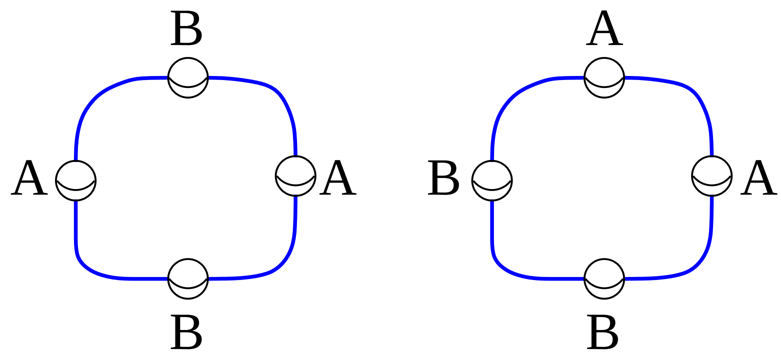

Figure 1. Kirby diagrams of the links $\mathbf{L}_1$ (left) and $\mathbf{L}_3$ (right) in $Y_2 := \#^2 S^1 \times S^2$. The S^2's (boundaries of the feet of 4–dimensional 1–handles) are identified as labeled, via a reflection in the plane perpendicular to the straight line joining their centers. The fibered link in each case is drawn in blue. To construct $\mathbf{L}_\ell \in Y_n$ in general, arrange n pairs of S^2's along an unknot in S^3 so that attaching 2–dimensional one-handles to the disk bounded by the unknot, via the chosen configuration, forms an oriented surface with ℓ boundary components.

Corollary 1. *If $Y \not\cong Y_n$ is a closed, oriented 3–manifold with the same Heegaard-Floer module structure as Y_n, then Y contains no fibered links of Euler characteristic $1-n$.*

There is a unique maximal Euler characteristic fibered link in S^3 (namely, the unknot) whose corresponding open book supports the standard tight contact structure. Ken Baker (cf. [8]) asked the following interesting question:

Question 1. Fix a contact structure, ξ, on a 3–manifold, Y, and let

$$\overline{\chi}_\xi := \max\{\chi(L) \mid L \text{ is a fibered link whose open book supports } \xi\}.$$

Up to diffeomorphism, are there finitely many fibered links L supporting ξ with $\chi(L) = \overline{\chi}_\xi$?

Proposition 2 tells us that for the standard tight contact structure on Y_n the answer is yes.

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1. Khovanov Homology Proof of Proposition 1

Proof of Proposition 1. Choose a diagram, $D(\widehat{\sigma})$, for $\widehat{\sigma}$ obtained as the closure of a diagram for σ, and mark the n points on the diagram corresponding to the intersection with the closure arc. Recall that the ($\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$) Khovanov homology, $\mathrm{Kh}(\widehat{\sigma})$, of $\widehat{\sigma}$ is an invariant of the isotopy class of $\widehat{\sigma} \subset S^3$ that takes the form of a bigraded vector space over $\mathbb{F}$. Since we have also chosen a basepoint on each of the n link components, [5, Prop. 1] tells us that

$\mathrm{Kh}(\widehat{\sigma})$ inherits the structure of a module over the ring

$$\mathcal{A}_n := \mathbb{F}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$$

as follows.

Associated to the diagram of $\widehat{\sigma}$ is a cube of resolutions whose vertices are in one-to-one correspondence with complete resolutions (i.e., Kauffman states) of the diagram. The basis elements (generators) of the underlying vector space of the Khovanov chain complex, $\mathrm{CKh}(D(\widehat{\sigma}))$, are, in turn, in one-to-one correspondence with markings of the components of each resolution with either a 1 or an x (i.e., *enhanced* Kauffman states).

Let $\mathcal{I}_{braid}$ be the unique "braid-like" complete resolution of $D(\widehat{\sigma})$, and denote by Ψ^+ (resp., Ψ^-) the basis element $1 \otimes \ldots \otimes 1$ (resp., $x \otimes \ldots \otimes x$) in the vector space associated to $\mathcal{I}_{braid}$. Ψ^- is a cycle, hence represents an element in $\mathrm{Kh}(\widehat{\sigma})$. Indeed, $[\Psi^-] \in \mathrm{Kh}(\widehat{\sigma})$ is precisely Plamenevskaya's invariant [14] of the transverse isotopy class of the transverse link represented by $\widehat{\sigma}$.

We are now ready to understand the $\mathcal{A}_n$ structure induced by the n points $p_1, \ldots, p_n$. For each complete resolution, $\mathcal{I}$, choose a numbering of its $\ell_{\mathcal{I}}$ connected components, and let $v_1 \otimes \ldots \otimes v_{\ell_{\mathcal{I}}}$ represent the Khovanov generator whose jth component in $\mathcal{I}$ is marked with $v_j \in \{1, x\}$. Suppose p_i lies on the kth component of $\mathcal{I}$. Then the action of $x_i \in \mathcal{A}_n$ is the $\mathbb{F}$–linear extension of the assignment:

$$x_i \cdot (v_1 \otimes \ldots \otimes v_k \otimes \ldots \otimes v_{\ell_{\mathcal{I}}}) := v_1 \otimes \ldots \otimes x \otimes \ldots \otimes v_{\ell_{\mathcal{I}}}$$

if $v_k = 1$ and 0 otherwise.

It is straightforward to check that the Khovanov differential commutes with the action of $\mathcal{A}_n$, and it is shown in [5] (see also [9], [10]) that the homotopy equivalences associated to Reidemeister moves respect the $\mathcal{A}_n$–module structure, and moving a basepoint past a crossing yields a homotopic map. The homology, $\mathrm{Kh}(\widehat{\sigma})$, therefore inherits the structure of an $\mathcal{A}_n$–module, and this $\mathcal{A}_n$–module structure is an invariant of the link.

With these preliminaries in place, assume that $\widehat{\sigma} = U_n$. A quick calculation using the standard diagram of U_n tells us that $\mathrm{Kh}(U_n) \cong \mathcal{A}_n$ as an $\mathcal{A}_n$-module. Let $\theta \in \mathrm{CKh}(D(\widehat{\sigma}))$ be a cycle representing the homology class $1 \in \mathrm{Kh}(U_n) \cong \mathcal{A}_n$.

We now claim that when θ is expressed as a linear combination of the standard Khovanov generators, the coefficient of Ψ^+ must be 1. To see this, note that $x_1 \cdots x_n(\theta)$ represents the non-zero homology class $x_1 \cdots x_n \in \mathrm{Kh}(\widehat{\sigma})$, but if v is any basis element not equal to Ψ^+, then $x_1 \cdots x_n(v) = 0$. We see this immediately for $v \neq \Psi^+ \in \mathcal{I}_{braid}$, and any complete resolution $\mathcal{I} \neq \mathcal{I}_{braid}$ contains at least one connected component intersecting the closure arc more than once, hence containing at least two basepoints p_i, p_j, $i \neq j$. We conclude that any basis element v associated to $\mathcal{I} \neq \mathcal{I}_{braid}$ satisfies $x_i x_j(v) = 0$, hence also satisfies $x_1 \cdots x_n(v) = 0$.

The arguments in the previous paragraph imply that $x_1 \cdots x_n(\theta) = x_1 \cdots x_n(\Psi^+) = \Psi^-$, so $[\Psi^-] = x_1 \cdots x_n \in Kh(\widehat{\sigma})$. In particular, $[\Psi^-] \neq 0$.

But [1, Prop. 3.1] then implies that σ is *right-veering.*

Repeat the argument above on $m(\sigma)$, the mirror of σ, to conclude that σ is also *left-veering.* Since the only braid which is both left- and right-veering is the identity braid (cf. [1, Lem. 3.1]), $\sigma = \mathbb{1}_n$, as desired.

$\square$

2. Fibred links in $\#^n(S^1 \times S^2)$

Recall that $\mathcal{L}_n := \{\ell \in \mathbb{Z}^+ \mid \ell \leq (n+1) \text{ and } \ell \equiv (n+1) \mod 2\}$.

Lemma 1. *If an ℓ–component link L has Euler characteristic $1-n$, then $\ell \in \mathcal{L}_n$.*

Proof. Let S denote the fiber surface of L, $\chi(S)$ its Euler characteristic, and $g(S)$ its genus. Then $\chi(S) = 1-n = (2-2g(S)) - \ell$. Since $g(S) \in \mathbb{Z}^{\geq 0}$, we obtain $\ell \equiv (n+1) \mod 2$ and $\ell \leq n+1$. □

Lemma 2. *If $L \subset Y_n$ is a fibered link, then $\chi(L) \leq 1-n$.*

Proof. Suppose L has ℓ components, and let S denote the fiber surface of L, and h its monodromy. $H_1(S)$ is free of rank $1-\chi(S) = 2g(S)+(\ell-1)$. Viewing $Y_n - L$ as the mapping torus of h (cf. Section 2.1), we obtain a corresponding presentation of $H_1(Y_n) \cong \mathbb{Z}^n$ with $1-\chi(L)$ generators, hence $1-\chi(L) \geq n$. □

2.1. **Heegaard-Floer homology proof of Proposition 2.** We begin with some background on Heegaard-Floer homology.

2.1.1. *Heegaard-Floer module.* Recall that in [12], Ozsváth-Szabó associate to a closed, oriented 3–manifold Y a graded vector space (for simplicity we work over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$), $\widehat{HF}(Y)$, which splits over $\mathrm{Spin}^c(Y)$, the set of spin^c structures on Y:

$$\widehat{HF}(Y) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \widehat{HF}(Y, \mathfrak{s})$$

For appropriate choices of symplectic and almost complex structures, $\widehat{HF}(Y)$ is the Lagrangian Floer homology of a natural pair of Lagrangian tori, $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$, in the g–fold symmetric product of a pointed Heegaard surface, (Σ, w), for Y.

$\widehat{HF}(Y)$ can be given the structure of a module over $\Lambda^*(H_1(Y;\mathbb{F}))$, as described in [12, Sec. 4.2.5]. Explicitly, let

$$(\Sigma, \alpha = \{\alpha_1, \ldots, \alpha_g\}, \beta = \{\beta_1, \ldots, \beta_g\}, z)$$

be a pointed, genus g Heegaard splitting of Y, and consider $\zeta \in H_1(Y;\mathbb{F})$. Ozsváth-Szabó define an associated chain map,

$$A_\zeta : \widehat{CF}(\Sigma, \alpha, \beta, z) \to \widehat{CF}(\Sigma, \alpha, \beta, z),$$

on the Heegaard-Floer chain complex as follows ([12, Rmk. 4.20]). Let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ be generators of the chain complex. Recall that $\pi_2(\mathbf{x}, \mathbf{y})$ denotes the set of domains in Σ representing topological Whitney disks connecting $\mathbf{x}$ to $\mathbf{y}$, in the sense of [12, Sec. 2.4]. If $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, we follow the notation in [11, Sec. 2.1], letting $\partial_\alpha \phi := (\partial\phi) \cap \mathbb{T}_\alpha$, regarded as a 1–chain with boundary $\mathbf{y} - \mathbf{x}$.

Choose an immersed curve,

$$\gamma_\zeta \subset \Sigma - \{\alpha_i \cap \beta_j\}_{i,j \in \{1,\ldots,g\}},$$

representing $\zeta \in H_1(Y;\mathbb{F})$ and define

$$a(\gamma_\zeta, \phi) := \#\widehat{\mathcal{M}}(\phi)(\gamma_\zeta \cdot \partial_\alpha \phi),$$

where $\gamma_\zeta \cdot \partial_\alpha \phi$ is the algebraic intersection number of γ_ζ and $\partial_\alpha \phi$. Then the chain map associated to ζ is given by:

$$A_\zeta(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x},\mathbf{y}) \mid \mu(\phi)=1, n_w(\phi)=0\}} a(\gamma_\zeta, \phi) \cdot \mathbf{y}.$$

The map A_ζ is well-defined (independent of the choice of γ) up to chain homotopy (cf. [11, Lem. 2.4]).

2.1.2. *Heegaard-Floer contact invariant.* We now recall the definition of the Heegaard-Floer contact invariant [13], following the alternative construction given in [7]. Let ξ be a contact structure on a closed, connected, oriented 3–manifold Y. Then Giroux tells us [4] that there exists some fibered link L whose corresponding open book supports ξ. One can then build a Heegaard diagram for $-Y$ (Y with the opposite orientation) using

- a choice of *basis*, $\{a_1, \ldots, a_n\}$, for a page S (of Euler characteristic $1-n$) of the open book [7, Sec. 3.1], and
- the data of the monodromy, h, of the open book.

Honda-Kazez-Matić then identify a distinguished cycle in the corresponding chain complex, $\widehat{CF}(-Y)$, and prove both that the class it represents in $\widehat{HF}(-Y)$ is invariant of the choices used in its construction and that it agrees with the contact invariant defined in [13].

We will need the following property of the contact invariant, which follows immediately from [13, Thm. 1.4] and [6, Thm. 1.1]:

Lemma 3. *If $L \subset Y$ is a fibered link whose monodromy, h, is not right-veering, then the Heegaard-Floer contact invariant associated to the contact structure supported by L is* 0.

We now proceed to the proof.

Proof of Proposition 2. Let $L_\ell \subset Y_n$ be an ℓ–component fibered link of Euler characteristic $1-n$. Construct a corresponding Heegaard diagram for $-Y_n$ as in [7, Sec. 3].

The module structure on $\widehat{HF}(-Y_n)$ has been computed in [12, Lem. 9.1]. Explicitly, $\widehat{HF}(-Y_n) \cong \mathcal{A}_n$ as a module over

$$\Lambda^*(H_1(-Y_n;\mathbb{F})) \cong \mathcal{A}_n := \mathbb{F}[\zeta_1, \ldots, \zeta_n]/(\zeta_1^2, \ldots, \zeta_n^2).$$

In particular, $\zeta_1 \cdots \zeta_n \neq 0 \in \widehat{HF}(-Y_n)$.

We can understand the module action explicitly in our setting as follows. All of our notation matches [7]. Examine the Honda-Kazez-Matić Heegaard diagram $\Sigma = S_{1/2} \cup -S_0$ associated to the fibered link, L_ℓ, and look at the "uninteresting" half, $S_{1/2} \subset \Sigma$, where the n–tuple of intersection points representing the contact class lives. Choose a compatible *dual basis*, $\{\gamma_1, \ldots, \gamma_n\}$, of simple closed curves on $S_{1/2}$ satisfying $|a_i \cap \gamma_j| = \delta_{ij}$. The set of homology classes, $\{[\gamma_1], \ldots, [\gamma_n]\}$, obtained by viewing the γ_i as 1–cycles in $-Y_n$, forms a basis for $H_1(-Y_n;\mathbb{F})$. Hence, for each $i \in \{1, \ldots, n\}$, the corresponding map on homology induced by the chain map $A_{[\gamma_i]}$ can be identified with $\zeta_i \in \mathcal{A}_n$.

Let $\theta \in \widehat{CF}(-Y_n)$ be any cycle representing $1 \in \widehat{HF}(-Y_n)$. Since $\zeta_1 \cdots \zeta_n \neq 0 \in \widehat{HF}(-Y_n)$, we know that there exists at least one generator $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ satisfying

$$\langle A_{[\gamma_1]} \cdots A_{[\gamma_n]} \cdot \theta, \mathbf{y} \rangle \equiv 1 \mod 2.$$

Associated to such a generator $\mathbf{y}$ is an odd number of corresponding Maslov index n domains in $\pi_2(\theta, \mathbf{y})$, each of which can be realized as the sum of n of the Maslov index 1 domains contributing to the chain maps $A_{[\gamma_1]}, \ldots, A_{[\gamma_n]}$. Consider the local multiplicity of such a Maslov index n domain, ψ, in the 4 regions adjacent to one of the constituent intersection points, x_i, of the distinguished cycle $\mathbf{x} = (x_1, \ldots, x_n)$ representing the contact class. We know (see Figure 2) that the local multiplicity of ψ in the two regions adjacent to x_i that contain the basepoint, z_0, must be 0 and also that the local multiplicity in the region adjacent to the unique intersection point between γ_i and a_i must be nonzero (hence positive, since ψ is a sum of domains representing holomorphic disks). Since the fourth region must have non-negative multiplicity, we conclude that x_i must be a corner, of multiplicity at least one, in the boundary of ψ, implying that x_i must be a constituent intersection point of the generator $\mathbf{y}$.

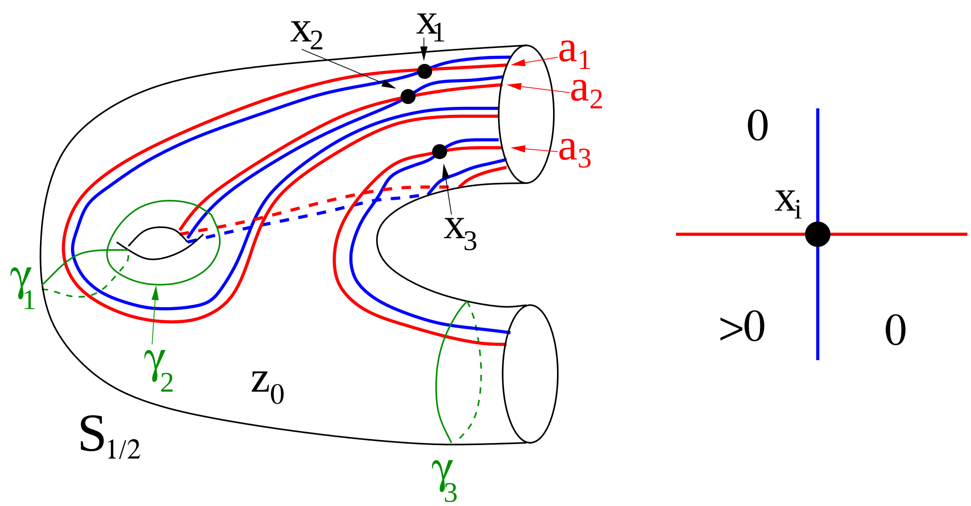

FIGURE 2. The "uninteresting" half of a Honda-Kazez-Matić Heegaard diagram associated to a fibered link $L_2 \subset Y_3$. The right-hand picture is a close-up of one of the constituent intersection points of the contact class and restrictions on the local multiplicities of the Maslov index n domain ψ. The NW, SE domains must have multiplicity 0 since they contain the basepoint z_0. One of the other two domains must have positive multiplicity, since it is the unique domain intersecting γ_i.

Since the above argument holds for each of the x_i, we conclude that, in fact, $\mathbf{y}$ is actually the distinguished contact class, $\mathbf{x}$, and it follows that (working mod 2) $A_{[\gamma_1]} \cdots A_{[\gamma_n]} \cdot \theta = \mathbf{x}$. Therefore,

$$[A_{[\gamma_1]} \cdots A_{[\gamma_n]} \cdot \theta] = [\mathbf{x}] = \zeta_1 \cdots \zeta_n \neq 0 \in \widehat{HF}(-Y_n),$$

so the Heegaard-Floer contact invariant associated to the contact structure supported by L_ℓ is nonzero. By Lemma 3, the monodromy, h, of L_ℓ is right-veering.

Now consider the *mirror* of L, i.e., the fibered link $L \subset -Y_n$ with monodromy h^{-1}. By running the same argument above, we conclude that the contact invariant associated to the contact structure supported by the mirror of L is also nonzero. Hence, h^{-1} is right-veering, implying that h is left-veering.

But if h is both right- and left-veering, then h is isotopic to the identity mapping class, and hence (Y_n, L_ℓ) is diffeomorphic as a pair to $(Y_n, \mathbf{L}_\ell)$. □

REFERENCES

[1] J.A. Baldwin and J.E. Grigsby. Categorified invariants and the braid group. math.GT/1212.2222, 2012.

[2] T. D. Cochran. Non-trivial links and plats with trivial Gassner matrices. *Math. Proc. Cambridge Philos. Soc.*, 119(1):43–53, 1996.

[3] P. Ghiggini and P. Lisca. Open book decompositions versus prime factorizations of closed, oriented 3–manifolds. math.GT/1407.2148, 2014.

[4] Emmanuel Giroux. Géométrie de contact: de la dimension trois vers les dimensions supérieures. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 405–414, Beijing, 2002. Higher Ed. Press.

[5] M. Hedden and Y. Ni. Khovanov module and the detection of unlinks. math.GT/1204.0960, 2012.

[6] K. Honda, W. H. Kazez, and G. Matić. Right-veering diffeomorphisms of compact surfaces with boundary. *Invent. Math.*, 169(2):427–449, 2007.

[7] Ko Honda, William H. Kazez, and Gordana Matić. On the contact class in Heegaard Floer homology. *J. Differential Geom.*, 83(2):289–311, 2009.

[8] J. Johnson. Heegaard splittings and open books. math.GT/1110.2142, 2011.

[9] M. Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000.

[10] M. Khovanov. Patterns in knot cohomology, I. *Experiment. Math.*, 12(3):365–374, 2003.

[11] Y. Ni. Homological actions on sutured Floer homology. math.GT/1010.2808, 2010.

[12] P. Ozsváth and Z. Szabó. Holomorphic disks and topological invariants for closed three-manifolds. *Ann. of Math. (2)*, 159(3):1027–1158, 2004.

[13] P. Ozsváth and Z. Szabó. Heegaard Floer homology and contact structures. *Duke Math. J.*, 129(1):39–61, 2005.

[14] O. Plamenevskaya. Transverse knots and Khovanov homology. *Math. Res. Lett.*, 13(4):571–586, 2006.

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