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A COMBINATORIAL PROOF OF THE HOMOLOGY COBORDISM CLASSIFICATION OF LENS SPACES

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Abstract. It follows implicitly from recent work in Heegaard Floer theory that lens spaces are homology cobordant exactly when they are oriented homeomorphic. We provide a new combinatorial proof using the Heegaard Floer d-invariants, which themselves may be defined combinatorially for lens spaces.

Introduction

An integer homology cobordism between two closed, oriented 3-manifolds Y_1 and Y_2 is a compact, oriented 4-manifold W whose boundary is $\partial W = Y_1 \cup -Y_2$ such that the inclusion maps induce isomorphisms $H_i(Y_1;\mathbb{Z}) \cong H_i(W;\mathbb{Z}) \cong H_i(Y_2;\mathbb{Z})$ for all homology groups; homology cobordism gives an equivalence relation. There are also corresponding definitions of rational homology cobordisms and spin-c rational homology cobordisms.

The homology cobordism classification of the lens spaces was only recently completed. In 1983, Gilmer and Livingston demonstrated that the lens spaces $L(p,q)$ for prime p are homology cobordant iff they are diffeomorphic [GL83]. Fintushel and Stern extended this result in 1988 for odd p [FS87]. Nicolaescu proved in 2001 that the Ozsváth-Szabó d-invariant recovers Reidemeister-Franz torsion [Nic04, Section 5], which, in turn, recovers homeomorphism type for lens spaces by results of Brody and Reidemeister [Bro60, Rei35] (technically, Nicolaescu showed the Ozsváth-Szabó theta divisor recovers the sum of the Casson-Walker invariant and Reidemeister-Franz torsion, but the Casson-Walker invariant of a lens space is the sum of its d-invariants, by a result of Rasmussen [Ras04, Lemma 2.2], and the theta divisor is the precursor of the d-invariant [OS]). In 2011, Greene showed that 2-bridge links are mutants iff their branched double covers (recall, all lens spaces are branched double covers of 2-bridge links) are homeomorphic iff the covers' $\widehat{HF}$ are the same [Gre13], but $\widehat{HF}$ recovers Reidemeister-Franz torsion by Rustamov [Rus, Theorem 3.4].

There are many known cobordism invariants, including some from Heegaard Floer homology. Ozsváth and Szabó associated the d-invariants to a manifold and spin-c structure which is invariant under spin-c rational homology cobordism, and the d-invariant function on the torsor of spin-c structures is likewise invariant under rational or integral homology cobordism [OS03, Theorem 1.2]. We provide a combinatorial proof that two lens spaces $L(p,q_1)$ and $L(p,q_2)$ share the same d-invariant function precisely when they are oriented homeomorphic. Since the d-invariants are defined combinatorially for lens spaces, this produces a proof of the homology cobordism classification of lens spaces which is entirely combinatorial (modulo the proof that the d-invariants are spin-c homology cobordism invariants; in fact, there is a proof of this invariance for lens spaces which is combinatorial except for its use of Donaldson's Theorem [Gre13]).

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Theorem 1. *Two lens spaces are cobordant by an integral homology cobordism exactly when they are oriented homeomorphic.*

We begin with a review of facts about d-invariants and spin-c structures and their behavior under homology cobordism. We also define a type of relative d-invariant $f(s,n)$ which carries all the information we need about the d-invariants. Next, we show that, if $Spin^c(L(p,q_1))$ and $d(L(p,q_1),\cdot)$ are isomorphic to $Spin^c(L(p,q_2))$ and $d(L(p,q_2),\cdot)$ in the category of torsors and functions, then $q_1 = q_2$ or $q_1q_2 \equiv 1 \pmod{p}$. Finally, we derive a more explicit description of the d-invariants modulo $\mathbb{Z}$ in the special case where p is prime.

Notation

Throughout this paper, let $[a]_p$ denote a representative of the class in the interval $[0,p)$. Let $a \equiv_p b$ mean a and b are equivalent modulo p. Let a' denote the inverse of a (if it exists), so $aa' \equiv_p 1$.

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d-invariants and spin-c structures

Heegaard Floer homology assigns several flavors of invariants (including HF^∞ and HF^+) to a closed, connected, oriented 3-manifold and a choice of spin-c structure using a Heegaard decomposition of the manifold [OS04b, OS04a]. The generators come with a relative $\mathbb{Z}$-grading. A spin-c cobordism $(W,\mathfrak{s})$ from $(Y_1,\mathfrak{s}|_{Y_1})$ to $(Y_2,\mathfrak{s}|_{Y_2})$ produces a map

$$F^+_{W,\mathfrak{s}} : CF^+(Y_1,\mathfrak{s}|_{Y_1}) \longrightarrow CF^+(-Y_2,\mathfrak{s}|_{-Y_2})$$

which induces a relative grading between generators for the two manifolds:

$$gr(F^+_{W,\mathfrak{s}}(x)) - gr(x) = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3sign(W)}{4}. \tag{1}$$

For an appropriate choice of spin-c manifold, including a rational homology sphere with its unique spin-c structure, this grading shift allows a lift of the relative $\mathbb{Z}$-grading to an absolute $\mathbb{Q}$-grading by fixing a canonical grading for S^3 with its unique spin-c structure.

Derived from this absolute grading is the *correction term* or *d-invariant* $d(Y,\sigma)$, the minimal grading of any non-torsion element in $HF^+(Y,\sigma)$ inherited from $HF^\infty(Y,\sigma)$ [OS03]. It is invariant under spin-c rational homology cobordism (if W is a rational homology cobordism, then the right side of Equation (1) is 0 for both W and $-W$). The d-invariants, as a function on a torsor over $H^2(Y) \cong H_1(Y)$, is also invariant under integral homology cobordism in the following fashion:

Proposition 2. *If Y_1 and Y_2 are integrally homology cobordant, then $Spin^c(Y_1)$ and $d(Y_1,\cdot)$ are isomorphic to $Spin^c(Y_2)$ and $d(Y_2,\cdot)$ in the category of torsors and functions.*

Proof. Let W be the 4-manifold cobordism with $\partial W = Y_1 \cup -Y_2$.

$Spin^c(Y_i)$ is a torsor over $H^2(Y_i) \cong H_1(Y_i)$, and $Spin^c(W)$ s a torsor over $H^2(W) \cong H_2(W,\partial W)$. The long exact sequence for the pair $(W,\partial W)$ splits:

$$0 \longrightarrow H_2(W,\partial W) \xrightarrow{r_1^* - r_2^*} H_1(Y_1) \oplus H_1(-Y_2) \longrightarrow H_1(W) \longrightarrow 0.$$

where r_i is the restriction map to $L(p, q_i)$. This short sequence induces isomorphisms

$$H_1(Y_1) \xleftarrow{r_1^*} H_2(W, \partial W) \xrightarrow{r_2^*} H_1(Y_2)$$

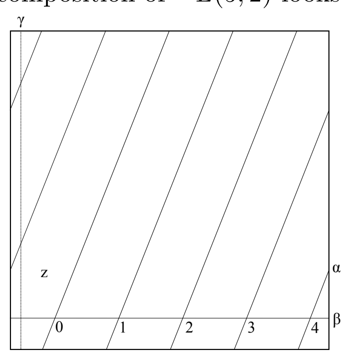
which in turn induce the required torsor isomorphism

$$Spin^c(Y_1) \xrightarrow{r_2 r_1^{-1}} Spin^c(Y_2).$$

There is a $\mathbb{Z}/2Z$ conjugation action $\mathfrak{t} \mapsto \bar{\mathfrak{t}}$ on the spin-c structures which fixes the spin structures. The restrictions maps and so also this isomorphism respect it.

Because it is invariant under spin-c homology cobordism, $d(Y_1, r_1(t)) = d(Y_2, r_2(t))$, and the functions $d(Y_i, \cdot)$ are isomorphic. $\square$

The lens space $-L(p, q)$ has a pointed Heegaard diagram (T^2, α, β, z) with a single α curve and β curve and exactly p intersection points $\alpha \cap \beta$, one in each of the p spin-c structures. For example, the Heegaard decomposition of $-L(5, 2)$ looks like:



We have chosen the orientation on $L(p, q)$ so that the manifold is $-p/q$ surgery on the unknot. Choose an identification of $Spin^c(L(p, q))$ by labelling the intersection points $0, 1, \ldots, p-1$ from left to right across the bottom of the diagram, beginning with the 0 for the bottom right corner of the domain containing the basepoint z [OS03, Proposition 4.8]. To see the difference of two spin-c structures $i - j \in H_1(L(p, q))$ under this identification, observe the curve γ, which is a generator of $H_1(L(p, q))$ and connects i to $i + q$ along the α curve and $i + q$ to i along the β curve, so we say $(i + q) - i = [\gamma]$. Any other $i - j$ gives a multiple of $[\gamma]$.

There is a combinatorial description of the d-invariants of a lens space based on the grading shift in Equation (1) derived in [OS03, Proposition 4.8]. Assuming $0 < q < p$,

$$d(L(p, q), i) = \frac{1}{4} - \frac{(2[i]_p + 1 - p - q)^2}{4pq} - d(L(q, p), i).$$

Derived from this recursive formula is a more direct formula for how the d-invariants change under the γ-action [LL08, Corollary 5.2]:

$$d(L(p, q), i + q) - d(L(p, q), i) = \frac{p - 1 - 2[i]_p}{p}. \tag{2}$$

The spin structures are exactly the integers among the following:

$$\frac{q-1}{2} \quad \text{and} \quad \frac{p+q-1}{2}. \tag{3}$$

This result may be deduced from Equation (2): The conjugation action which fixes a spin structure s must identify $s + n$ and $s - n$, and $d(L(p, q), i + q) = d(L(p, q), i - q)$ implies $\frac{p-1-2i}{p} = -\frac{p-1-2(i-q)}{p}$, or $2i \equiv_p q - 1$. For alternative explanations, Cf [Ue09, p. 134] or

[CH15, Lemma 6.1]. Note that both numbers in (3) are axes of symmetry, and both are integers when p is even, but only one is an integer (and so a spin structure) when p is odd.

In the case of two homology cobordant lens spaces $L(p, q_i)$, Proposition 2 also implies:

$$d(L(p,q_1), s_1 + a) = d(L(p,q_2), r_2 r_1^{-1}(s_1) + (r_2 r_1^{-1})^*(a)) = d(s_2 + ua).$$

for any $a \in H_1(L(p, q_1))$, where s_1 and s_2 are chosen spin structures which are restrictions of a common spin structure on W. The last equality follows because $(r_2 r_1^{-1})^*$ is an isomorphism of $\mathbb{Z}/p\mathbb{Z}$, which means it is multiplication by some unit $u \in \mathbb{Z}/p\mathbb{Z}$.

Relative d-invariants

Let p and q be coprime with $p > q > 0$. Choose a spin structure s as in (3) (if p is odd, this choice is forced). We renormalize the d-invariants of $L(p, q)$ by defining a function $f(s, \cdot)\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}$ using this choice of spin structure:

$$f(s, n) := pd(L(p, q), s + nq) - pd(L(p, q), s).$$

Lemma 3. *The function f obeys*

$$\begin{aligned} f(s,0) &= 0 \\ f(s,n+1) &= f(s,n) + p - 1 - 2\,[s + nq]_p \\ f(s,n) &\equiv_p -n^2 q \end{aligned}$$

If $L(p, q_1)$ and $L(p, q_2)$ are homology cobordant by W, and if f_1 and f_2 are the corresponding functions for some compatible choice of spin structure s_1 and s_2 which restrict the same spin structure on W,

$$\begin{aligned} f_2(s_2, n) &= f_1(s_1, nu) \\ q_2 &\equiv_p u^2 q_1 \end{aligned}$$

for some unit $u \in \mathbb{Z}/p\mathbb{Z}$.

A cobordism between two lens spaces tells us about the torsor structure defined above.

Proof. The first two equalities follow from Equation (2) and the definitions of f and s. The third equality holds because $f(s, n+1) \equiv_p f(s, n) - (2n+1)q$ and $f(s, 0) \equiv_p 0$. The fourth follows from Proposition 2, assuming that the spin structures were chosen so that $r_2 r_1^{-1}(s_1) = s_2$, and the last equality follows from the third and fourth. □

Proof of Theorem 1

We will now prove the main theorem.

Proof of Theorem 1. By Lemma 3, there is a unit u such that

$$q_1 = q \quad \text{and} \quad q_2 \equiv_p u^2 q.$$

There is also some choice of spin structures s_1 and s_2 which are restrictions of the same spin structure on W. Define $g\colon \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}$ such that

$$g(m) := f_1(s_1, mq') = f_2(s_2, mu'q'),$$

which is well-defined by the definitions of $f_i(s_i, m)$ and Lemma 3.

Apply the recursive equations for $f_i(s_i, n)$ from Lemma 3 to $f_1(s_1, mq'{+}1)$ and $f_2(s_2, mu'q'+ 1)$ to see that g satisfies the relations:

$$g(m + q) = g(m) + p - 1 - 2\,[s_1 + m]_p$$

and

$$g(m+uq) = g(m) + p - 1 - 2\left[s_2 + mu\right]_p.$$

Since the above relations must hold for all m, we can compute $g(m+uq+q)$ in two ways, as $g((m+uq)+u)$ or as $g((m+u)+uq)$. Since the results must be the same, we get

$$\left[s_1 + m\right]_p + \left[s_2 + (m+q)u\right]_p = \left[s_2 + mu\right]_p + \left[s_1 + m + uq\right]_p. \tag{4}$$

Now recall that

$$[X+Y]_p = \begin{cases} [X]_p + [Y]_p & \text{if } [X]_p < p - [Y]_p, \\ [X]_p + [Y]_p - p & \text{if } [X]_p \geq p - [Y]_p. \end{cases}$$

Equation (4) is therefore equivalent to the condition that

$$\left[s_1 + m\right]_p < p - \left[uq\right]_p \iff \left[s_2 + mu\right]_p < p - \left[uq\right]_p \tag{5}$$

for all $m \in \mathbb{Z}/p\mathbb{Z}$.

By Lemma 5 below, Condition (5) can only be satisfied for all $m \in \mathbb{Z}/p\mathbb{Z}$ if either $u \equiv_p \pm 1$ or $uq \equiv_p \pm 1$. That is, either $q_2 = q_1$ or $q_1 q_2 \equiv_p 1$. □

Note that we did not use any information in the above proof about the explicit forms the s_i take, merely the fact that there exist s_1 and s_2 which are restrictions of some spin structure on W; in particular, the parity of p is irrelevant.

We now address two technical lemmata required for the proof above.

Lemma 4. *Let*

$$H : \{0, 1, \cdots, p-1\} \longrightarrow \{0, 1, \cdots, p-1\}$$

be a function such that $H(i) \equiv_p H(0) + in$*. If*

$$H(i) < C \iff i < C$$

where $2 \leq C \leq p-2$*, then*

$$H(i) = i \quad \text{or} \quad H(i) = C - 1 - i.$$

A few experiments will quickly convince the reader that this lemma should be true. For the sake of completeness, we prove:

Proof. Choose $-p/2 \leq n \leq p/2$. Assume, for the moment, that $C \leq p/2$.

If $n = 1$, then $H(0) = 0$ and $H(i) = i$.

If $n = -1$, then $H(0) = C - 1$ and $H(i) = C - i - 1$.

For any other n, there will eventually be an $i < C$ with $H(i) \geq C$. For example, for $n \geq 2$, take

$$i_0 = \left\lfloor \frac{C - H(0)}{n} \right\rfloor + 1$$

Note that $0 < i_0 < C$ since $C \geq 2$, and

$$0 < H(0) + ni_0 \leq H(0) + n\left(\frac{C - H(0)}{n} + 1\right) = C + n \leq p$$

so we may remove the $\equiv_p$ in the definition of $H(i_0)$:

$$H(i_0) = H(0) + ni_0 \geq H(0) + n\left(\frac{C - H(0)}{n}\right) = C,$$

as desired.

Similarly, for $n \leq -2$, take

$$i_0 = \left\lfloor \frac{H(0)}{|n|} \right\rfloor + 1.$$

Now $-p < H(0) + ni_0 < 0$, so

$$H(i_0) = H(0) + ni_0 + p \geq H(0) - |n| \left(\frac{H(0)}{|n|} + 1 \right) + p \geq p - |n| \geq p/2 \geq C$$

It is easy to adjust the above proof to accommodate $C \geq p/2$. The key is that some (at least two and at most $p-2$) adjacent values of i map to (the same number of) adjacent values of $H(i)$. The following are equivalent:

$$\begin{aligned}
H(i) < C &\iff i < C \\
0 \leq H(i) \leq C - 1 &\iff 0 \leq i \leq C - 1 \\
p - C \leq H(i) + p - C \leq p - 1 &\iff p - C \leq i + p - C \leq p - 1 \\
p - C \leq H(i - p + C) + p - C \leq p - 1 &\iff p - C \leq i \leq p - 1 \\
H(i + C) - C < p - C &\iff i < p - C
\end{aligned}$$

and $\widehat{H}(i) = H(i + C) - C$ also obeys the rule $\widehat{H}(i) \equiv_p \widehat{H}(0) + in$. $\square$

Lemma 5. *Let*

$$\begin{aligned}
f(m) &= [x + my]_p \\
F(m) &= [X + mY]_p.
\end{aligned}$$

with y and Y units modulo p. If

$$f(m) < C \iff F(m) < C$$

for some $2 \leq C \leq p - 2$, then

$$Y = \pm y.$$

Proof. Rescale m by precomposing f and F with

$$m(i) = (i - x)y'.$$

Then

$$h(i) := f(m(i)) = [i]_p$$

and

$$H(i) := F(m(i)) = [(X - xy'Y) + i(y'Y)]_p.$$

The lemma statement is equivalent to

$$h(i) < C \iff H(i) < C,$$

which is equivalent to

$$H(i) < C \iff i < C.$$

Note that $H(i) \equiv_p H(0) + in$, and apply Lemma 4.

If $H(i) = i$, then $y'Y = 1$, or $Y = y$, and $X - xy'Y = 0$, or $X = x$.

If $H(i) = C - 1 - i$, then $H(0) = C - 1 \equiv_p X - xy'Y$ and $H(C-1) = 0 \equiv_p X - xy'Y + (C-1)y'Y \equiv_p C - 1 + (C-1)y'Y$, so $Y \equiv_p -y$ and $X \equiv_p -x + C - 1$. $\square$

If p is prime

In the special case where p is a prime, we have a more precise description of the d-invariants modulo $\mathbb{Z}$. Consider the reduction of f modulo p, $\overline{f}(s,\cdot)\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$. We denote by

$$\overline{S}(L(p,q)) \subseteq \mathbb{Z}/p\mathbb{Z}$$

the image of $\overline{f}$.

Theorem 6. *Let p be prime number and q coprime to p.*

(a) If q is a quadratic residue modulo p, then

$$\overline{S}(L(p,q)) = \{a \in \mathbb{Z}/p\mathbb{Z} \,|\, -a \textit{ is a square in } \mathbb{Z}/p\mathbb{Z}\}.$$

(b) If q is a quadratic non-residue modulo p, then

$$\overline{S}(L(p,q)) = \{a \in \mathbb{Z}/p\mathbb{Z} \,|\, -a \textit{ is not a square in } \mathbb{Z}/p\mathbb{Z}\} \cup \{0\}.$$

In the residue case, a more explicit description of the d-invariants is possible:

Corollary 7. *Let p be an odd prime number and q a residue coprime to p.*

(a) There is only one n such that $\overline{f}(s,n) = 0$, namely, $n = 0$.
(b) For every $a \in \overline{S}(L(p,q)) \setminus \{0\}$, there are exactly two n such that $\overline{f}(s,n) = a$.
(c) $\overline{S}(L(p,q))$ contains exactly $(p+1)/2$ elements.

Proof of Theorem 6. Since $f(s,n) \equiv_p -n^2 q$,

$$\overline{S}(L(p,q)) = \{a \in \mathbb{Z}/p\mathbb{Z} \,|\, a \text{ satisfies } a \equiv_p -n^2 q \text{ for some } n\}.$$

If $a = 0$, then $n = 0$.

Let $\left(\frac{m}{p}\right)$ denote the Legendre symbol of m and p, defined by

$$\left(\frac{m}{p}\right) := \begin{cases} 1 & \text{if } m \text{ is a quadratic residue modulo } p, \\ -1 & \text{if } m \text{ is a quadratic non-residue modulo } p, \\ 0 & \text{if } m \text{ is zero modulo } p. \end{cases}$$

Assume $a \neq 0$. Then the condition $a \equiv_p -n^2 q$ can be written as $-aq' \equiv_p n^2$, or

$$\left(\frac{-aq'}{p}\right) = 1.$$

Since the Legendre symbol is multiplicative in the first argument, we can write the condition as

$$\left(\frac{-a}{p}\right)\left(\frac{q'}{p}\right) = 1,$$

and, multiplying both sides by $\left(\frac{q}{p}\right)$, we get

$$\left(\frac{-a}{p}\right) = \left(\frac{q}{p}\right),$$

where we have used that $\left(\frac{q'}{p}\right)\left(\frac{q}{p}\right) = \left(\frac{qq'}{p}\right) = \left(\frac{1}{p}\right) = 1$. We can thus write $\overline{S}(L(p,q))$ as

$$\overline{S}(L(p,q)) = \left\{ a \in \mathbb{Z}/p\mathbb{Z} \,\middle|\, a = 0 \text{ or } \left(\frac{-a}{p}\right) = \left(\frac{q}{p}\right) \right\}.$$

□

Proof of Corollary 7. If p is prime, $(\mathbb{Z}/p\mathbb{Z})[x]$ is a unique factorization domain. If $p \neq 2$, this means every equation $n^2 \equiv_p -aq'$ with $a \neq 0$ has exactly two solutions. Part (c) follows because the total number of d-invariants, counted with multiplicities, is equal to p. $\square$

Note that $\overline{S}(L(2,1)) = \mathbb{Z}/2\mathbb{Z}$, so (b) and (c) are false for $p = 2$.

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