MAT 762, Algebraic Topology, Fall 2019

Homework Assignment 1

Problem 1. A topological group is a topological space G with a group structure such that both the multiplication map $G \times G \to G$ and the inversion map $G \to G$, $g \mapsto g^{-1}$, are continuous. Let G be a topological group with identity element $e \in G$.

(a) If $\gamma, \eta \colon [0,1] \to G$ are loops based at e, then γ and η can be composed in two different ways: either by using the usual composition of paths defined in Hatcher on page 26, or by using the composition defined by $(\gamma \odot \eta)(s) := \gamma(s)\eta(s)$ $\forall s \in [0,1]$ where the product on the right-hand side is taken in the group G. Show that $[\gamma \cdot \eta] = [\gamma \odot \eta] \in \pi_1(G, e)$.

(b) Show that $\pi_1(G, e)$ is abelian.

(c) Which of the following surfaces carry/carries a topological group structure?

  (i) 2-Sphere.

  (ii) Torus.

  (iii) Closed genus 2 surface.

In (c), no detailed justification is required.

Problem 2. Show that $\bigwedge(V \oplus W) \cong \bigwedge V \hat{\otimes} \bigwedge W$, where $\hat{\otimes}$ denotes the graded-commutative tensor product of graded-commutative graded algebras: it is the same tensor product additively, but the algebra multiplication satisfies $(a \otimes b) \cdot (c \otimes d) = (-1)^{\deg(b)\deg(c)}(a \cdot c) \otimes (b \cdot d)$ for homogeneous elements a and c in the first algebra, and homogeneous elements b and d in the second.

Problem 3. Suppose $\{G_i, f_{ij}\}$, $\{G'_i, f'_{ij}\}$, $\{G''_i, f''_{ij}\}$ are direct systems of abelian groups over an index set I, and suppose $f_i \colon G_i \to G'_i$ and $g_i \colon G'_i \to G''_i$ are group homomorphisms such that $f_j \circ f_{ij} = f'_{ij} \circ f_i$ and $g_j \circ f'_{ij} = f''_{ij} \circ g_i$ for all pairs $i \leq j$, and such that the sequence

$$G_i \xrightarrow{f_i} G'_i \xrightarrow{g_i} G''_i$$

is exact for all i (i.e., $\operatorname{im}(f_i) = \ker(g_i)$). Prove that there is an induced sequence

$$\varinjlim G_i \xrightarrow{f} \varinjlim G'_i \xrightarrow{g} \varinjlim G''_i$$

which is also exact.

This homework is due on Tuesday, September 10, 2019.