MAT 762, Algebraic Topology, Fall 2019

Homework Assignment 2

Problem 1. Recall that the projective plane $\mathbb{R}\mathrm{P}^2$ can be obtained from S^1 by attaching a disk D^2 via a degree 2 map $\partial D^2 \to S^1$. In particular, there is a quotient map $f\colon \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2/S^1 \cong S^2$.

(a) Give S^1 its usual cell structure with just two cells and write down the cellular chain complexes for $\mathbb{R}\mathrm{P}^2$ and $S^2 \cong \mathbb{R}\mathrm{P}^2/S^1$, as well as the chain map $f_\#$ between these two chain complexes.

(b) Compute the map $f_*\colon H_*(\mathbb{R}\mathrm{P}^2) \to H_*(S^2)$.

(c) Repeat parts (a) and (b) for coefficients in $\mathbb{Z}_2$.

(d) Conclude that there can't be a natural isomorphism

$$H_2(-;\mathbb{Z}_2) \cong H_2(-) \otimes \mathbb{Z}_2 \oplus \mathrm{Tor}(H_1(-), \mathbb{Z}_2)$$

where natural would mean that for any two spaces X and Y and any continuous map $f\colon X \to Y$, the map that f induces on the left-hand side agrees with the direct sum of the maps that f induces on the summands on the right-hand side.

Problem 2. Fix $n > 0$ and let Y be the quotient space $\mathbb{R}^n/(\mathbb{R}^n - \{0\})$.

(a) Describe the topological space Y explicitly.

(b) Show that Y is contractible.

(c) What is the reduced homology of Y? Is it isomorphic to $H_*(\mathbb{R}^n, \mathbb{R}^n - \{0\})$?

Problem 3. Let X be the space obtained from S^1 by attaching a disk D^2 via the map $\partial D^2 \to S^1$, $z \mapsto z^m$, for $m > 2$.

(a) Compute the cellular homology of X for coefficients in $\mathbb{Q}$.

(b) Show that for any point $x \in S^1$, the space $X - \{x\}$ deformation retracts to a bipartite graph with 2 vertices and m edges (no rigorous proof is required).

(c) Use (a) and (b) to compute the second local homology of X for coefficients in $\mathbb{Q}$ at a point $x \in S^1$.

(d) Is X a 2-manifold?

This homework is due on Thursday, September 19, 2019.