MAT 762, Algebraic Topology, Fall 2019

Homework Assignment 3

Problem 1. Let $\mathbb{R}^\infty$ denote the space of all sequences $x = (x_1, x_2, \ldots)$ of real numbers such that $x_i \neq 0$ for only finitely many i. Further, let $S^\infty \subset \mathbb{R}^\infty$ denote the unit sphere with respect to the Euclidean norm on $\mathbb{R}^\infty$. The sphere S^∞ has a cell decomposition

$$S^\infty = (e^0_+ \cup e^0_-) \cup (e^1_+ \cup e^1_-) \cup (e^2_+ \cup e^2_-) \cup \ldots$$

where e^n_+ and e^n_- are two n-cells given respectively by the upper ($x_{n+1} > 0$) and the lower ($x_{n+1} < 0$) hemisphere of $S^n = S^\infty \cap \{x_i = 0 \,|\, i > n+1\}$. Assume now that S^∞ is equipped with the topology induced by this cell decomposition, and let $\tau \colon S^\infty \to S^\infty$ denote the antipodal map. If we choose the characteristic maps of $e^n_\pm$ in such a way that $\Phi^n_- = \tau \circ \Phi^n_+$ and that e^{n-1}_+ appears with coefficient 1 in the cellular boundary of e^n_+, then

$$\partial e^n_+ = e^{n-1}_+ + s_n e^{n-1}_- = (\mathrm{id} + s_n \tau_\#)(e^{n-1}_+)$$

for a sign $s_n = \deg(\tau \colon S^{n-1} \to S^{n-1}) \in \{\pm 1\}$.

(a) Use $\partial e^1_+ = e^0_+ - e^0_-$; $\partial \circ \partial = 0$; and $\tau_\# \circ \partial = \partial \circ \tau_\#$ to determine s_n for all $n > 0$.

(b) Use part (a) to compute the cellular homology of S^∞.

(c) Use part (a) to compute the cellular chain complex of $\mathbb{R}\mathrm{P}^\infty = S^\infty / \{x \sim \tau(x)\}$.

Problem 2. (Hatcher, Section 3.1, Exercise 5) Regarding a cochain $\varphi \in C^1(X)$ as a function from paths in X to $\mathbb{Z}$, show that if φ is a cocycle, then

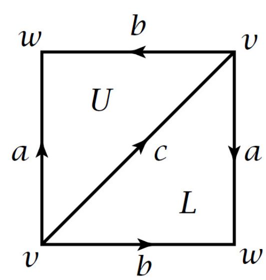
(a) $\varphi(\gamma \cdot \eta) = \varphi(\gamma) + \varphi(\eta)$,

(b) φ takes the value 0 on constant paths,

(c) $\varphi(\gamma) = \varphi(\eta)$ if γ is homotopic to η with fixed endpoints,

(d) φ is a coboundary iff $\varphi(\gamma)$ depends only on the endpoints of γ, for all γ.

Problem 3. Let X be the following Δ-complex:



(a) Compute the simplicial homology of X.

(b) Compute the simplicial cohomology of X in two ways: first directly and then by using part (a) and the universal coefficient theorem for cohomology.

This homework is due on Tuesday, October 1, 2019.