MAT 762, Algebraic Topology, Fall 2019

Homework Assignment 6

Problem 1. Let $n > 0$, and suppose that the n-sphere S^n carries an H-space structure. That is, suppose there is a continuous map $\mu\colon S^n \times S^n \to S^n$ and a point $e \in S^n$ such that the maps $x \mapsto \mu(x,e)$ and $x \mapsto \mu(e,x)$ are homotopic to the identity map of S^n. Show the following:

(a) $\mu \circ i_1$ and $\mu \circ i_2$ induce the identity on $H^n(S^n)$, where $i_1, i_2\colon S^n \to S^n \times S^n$ are the maps $x \mapsto (x,e)$ and $x \mapsto (e,x)$.

(b) $p_1 \circ i_1$ and $p_2 \circ i_2$ induce the identity on $H^n(S^n)$, where $p_1, p_2\colon S^n \times S^n \to S^n$ are the projections onto the two factors.

(c) $p_2 \circ i_1$ and $p_1 \circ i_2$ induce the zero map on $H^n(S^n)$.

Problem 2. (Problem 1, continued) Using the Künneth Theorem, one can easily see that $H^n(S^n \times S^n)$ is freely generated by $\alpha_1 := p_1^*(\alpha) = \alpha \times \epsilon$ and $\alpha_2 := p_2^*(\alpha) = \epsilon \times \alpha$, and $H^{2n}(S^n \times S^n)$ is freely generated by $\alpha_1 \cup \alpha_2 = \alpha \times \alpha$, where α denotes a generator of $H^n(S^n)$ and ϵ denotes the identity element in the cohomology ring of S^n.

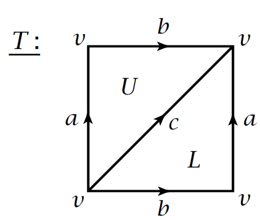
(a) Write $\mu^*(\alpha)$ as a linear combination $n_1\alpha_1 + n_2\alpha_2$ and use the previous problem to show that $n_1 = 1 = n_2$. *Hint.* Apply i_j^* to both sides of $\mu^*(\alpha) = n_1\alpha_1 + n_2\alpha_2$.

(b) Conclude that $(\alpha_1 + \alpha_2) \cup (\alpha_1 + \alpha_2) = 0$.

(c) Conclude that n has to be odd.

Problem 3. (Hatcher, Section 3.2, Exercise 11) Using cup products, show that every map $S^{k+\ell} \to S^k \times S^\ell$ induces the trivial homomorphism $H_{k+\ell}(S^{k+\ell}) \to H_{k+\ell}(S^k \times S^\ell)$, assuming $k > 0$ and $\ell > 0$. *Hint.* It suffices to show that the induced homomorphism $H^{k+\ell}(S^k \times S^\ell) \to H^{k+\ell}(S^{k+\ell})$ is trivial.

Problem 4. As on Homework Assignment 4, let T be the Δ-complex shown below:



Recall that $H_*(T)$ has a basis given by $\{[v],[a],[b],[L-U]\}$ and $H^*(T)$ has a basis given by $\{[v^*],[a^*+c^*],[b^*+c^*],[L^*]\}$. For each α in the latter basis, compute the cap product $[L-U] \cap \alpha$ explicitly, and conclude that $[L-U] \cap -$ is an isomorphism from $H^*(T)$ to $H_*(T)$.

This homework is due on Thursday, October 31, 2019.