MAT 762, Algebraic Topology, Fall 2019

Homework Assignment 7

Problem 1. For a continuous map $f\colon M \to N$ between connected closed orientable n-manifolds, suppose there is a point $y \in N$ such that $f^{-1}(y)$ consists of a single point $x \in M$. Show that if there is an open neighborhood $U \subset M$ of x such that f maps U homeomorphically to an open neighborhood $V \subset N$ of y, then f induces an isomorphism $f_*\colon H_n(M) \to H_n(N)$. *Hint.* Use that there are isomorphisms $H_n(M) \cong H_n(M|x) \cong H_n(U|x)$ and $H_n(N) \cong H_n(N|y) \cong H_n(V|y)$.

Problem 2. For a connected closed orientable n-manifold M, show that there is a continuous map $f\colon M \to S^n$ which induces an isomorphism $f_*\colon H_n(M) \to H_n(S^n)$. *Hint.* Choose a Euclidean neighborhood $U \subset M$ and use that it can be identified with $S^n - \{p\}$ for $p \in S^n$.

Problem 3. A subset $J \subset I$ of a directed poset $(I, \geq)$ is called *cofinal* if for every $i \in I$ there is a $j \in J$ such that $j \geq i$. Show that if $\{G_i, f_{ii'}\}$ is a direct system of abelian groups over a directed poset $(I, \geq)$ and $J \subset I$ is a cofinal subset, then J is itself directed and $\varinjlim_{j\in J} G_j = \varinjlim_{i \in I} G_i$.

Problem 4. Let Y be a compact Hausdorff space and let X be the space $X := Y - \{y\}$ for a point $y \in Y$. Show the following:

(a) If $U \subset Y$ is an open neighborhood of y which deformation retracts to y, then the inclusions $(Y, y) \hookrightarrow (Y, U) \hookleftarrow (X, U \cap X)$ induce isomorphisms in relative cohomology.

(b) If there is a neighborhood basis at $y \in Y$ consisting of open neighborhoods of y which deformation retract to y, then $H^n(Y, y) \cong \varinjlim_{K\in I} H^n(X, X - K)$ where I is the set of all compact subspaces $K \subset X$, ordered so that $K \leq K'$ iff $K \subset K'$.

This homework is due on Tuesday, November 12, 2019.