MAT 762, Algebraic Topology, Fall 2019

Homework Assignment 8

Problem 1. A *symplectic form* on a vector space V is a bilinear form ω on V satisfying $\omega(v,v) = 0$ for all $v \in V$ and such that for every $v \in V$, there is a $w \in V$ with $\omega(v,w) \neq 0$. A *symplectic vector space* is a vector space V equipped with a symplectic form ω. Show the following:

(a) If V is a nonzero symplectic vector space, then there are linearly independent vectors $v, w \in V$ such that $\omega(v,w) = 1$.

(b) If v, w are as in part (a) and $U \subset V$ is the span of v, w, then $V = U \oplus U^\perp$ where $U^\perp := \{x \in V \mid \omega(x,u) = 0 \text{ for all } u \in U\}$.

(c) If U is as in part (b), then the restriction of ω to $U^\perp$ is itself symplectic.

(d) Every finite-dimensional symplectic vector space is even-dimensional.

Hint. You can prove (b) by showing that the map $p\colon V \to V$ given by $p(x) := \omega(x,w)v - \omega(x,v)w$ is a projection onto U with kernel $U^\perp$.

Problem 2. Let n be odd and M be a closed orientable $2n$-manifold. For a field F of characteristic zero, show the following:

(a) The intersection pairing $\omega\colon H_n(M;F) \otimes H_n(M;F) \to F$ given by $\omega(a,b) := \langle \alpha \cup \beta, [M] \rangle = \langle \beta, [M] \cap \alpha \rangle = \langle \beta, a \rangle = \beta(a)$ for $\alpha := D_M^{-1}(a)$ and $\beta := D_M^{-1}(b)$ is a symplectic.

(b) $H_n(M;F)$ is even-dimensional.

Hint. In (b), you can use without proof that $H_n(M;F)$ is finite-dimensional.

Problem 3. For a continuous map $f\colon M \to N$ between connected closed oriented n-manifolds with fundamental classes $[M]$ and $[N]$, the *degree* of f is defined to be the integer d such that $f_*([M]) = d[N]$. Note that the degree is multiplicative under composition of continuous maps, by construction. Show the following:

(a) If f is a homeomorphism, then $\deg(f) = \pm 1$.

(b) If $f\colon S^1 \to S^1$ is the map given by $z \mapsto z^d$ for $z \in S^1 \subset \mathbb{C}$ and an integer $d \neq 0$, then $\deg(f) = \pm d$.

(c) If $f\colon S^1 \times S^1 \to S^1 \times S^1$ is the map given by $(z,w) \mapsto (z^{d_1}, w^{d_2})$ for integers $d_1, d_2 \neq 0$, then $\deg(f) = \pm d_1 d_2$.

(d) If $f_A\colon S^1 \times S^1 \to S^1 \times S^1$ is the map given by $(z,w) \mapsto (z^{a_{11}} w^{a_{12}}, z^{a_{21}} w^{a_{22}})$ for a 2×2 matrix A with entries $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{Z}$ and $\det(A) \neq 0$, then $\deg(f_A) = \pm \det(A)$.

Hint. To prove (c), you can use (b) and the Künneth Theorem. To deduce (d) from (c), you can use without proof that there are 2×2 matrices B and C with integer entries and with determinant ± 1, such that BAC is diagonal. You can further use that $f_{A'} \circ f_A = f_{A'A}$.

This homework is due on Thursday, November 21, 2019.